Natural duality for default bilattices

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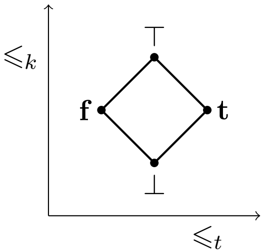
Outline

- Bilattices
- Default bilattices
- Natural duality theory
- Natural duality for quasivarieties of default bilattices
- Current and future work

Introduction

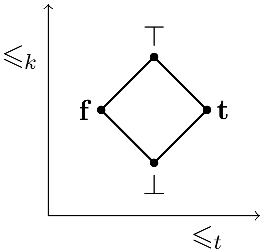
The structure shown below was introduced by Belnap in two papers in the mid-1970's ("A useful four-valued logic" and "How a computer should think").



In 1986, Ginsberg generalised Belnap's four-valued logic and defined what are now known as **bilattices**.

Bilattices

The vertical axis represents the **knowledge** order and the horizontal axis represents the **truth** order. The lattice operations of the knowledge order ($\otimes$ and $\oplus$) represent *consensus* and *gullibility*.



In addition to two sets of lattice operations, a bilattice has a unary negation operation $\neg$ which **preserves** $\leqslant_k$ and **reverses** $\leqslant_t$.

Note: The knowledge order is often called the information order. In the literature an algebra with two lattices structures but without a negation operation is called a pre-bilattice.

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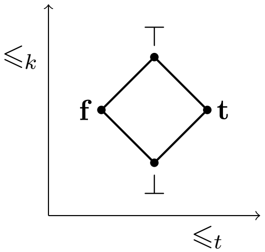

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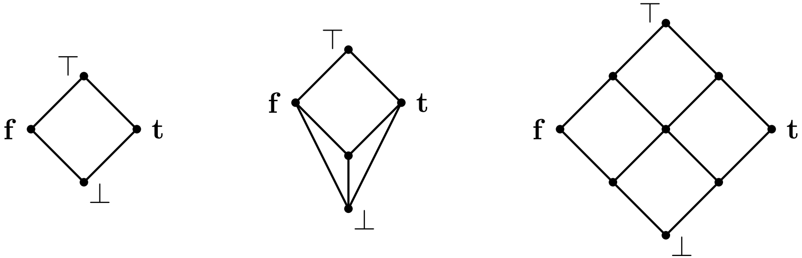
Examples of bilattices:



$\mathcal{FOUR}$ $\mathcal{FIVE}$ $\mathcal{NINE}$

For a bilattice $\mathbf{B}$ we have

$$\mathbf{B} = \langle B; \otimes, \oplus, \wedge, \vee, \neg \rangle.$$

We will often add bounds of one (or both) of the orders to the signature. The knowledge bounds are $\bot, \top$ and the truth bounds are $\mathbf{f}, \mathbf{t}$.

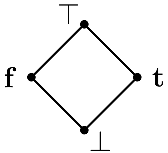
Distributive bilattices

There can be different levels of interaction between the orders. A bilattice $\mathbf{B}$ is **distributive** if for all $a, b, c \in B$ the identity

$$a \bullet (b * c) \approx (a \bullet b) * (a \bullet c)$$

holds for $\bullet, * \in \{\otimes, \oplus, \wedge, \vee\}$.

The variety of distributive bilattices is generated by $\mathcal{FOUR}$:



$$\mathscr{DB} = \mathbb{HSP}(\mathcal{FOUR}) = \mathbb{ISP}(\mathcal{FOUR}).$$

Interlaced bilattices

A bilattice is **interlaced** if the knowledge operations preserve the truth order and the truth operations preserve the knowledge order. That is,

$$\begin{aligned} a \leqslant_t b &\implies a \otimes c \leqslant_t b \otimes c, \\ a \leqslant_t b &\implies a \oplus c \leqslant_t b \oplus c, \\ a \leqslant_k b &\implies a \wedge c \leqslant_k b \wedge c, \\ a \leqslant_k b &\implies a \vee c \leqslant_k b \vee c. \end{aligned}$$

Note: an interlaced bilattice $\mathbf{B}$ has $\leqslant_k$ as a subalgebra of $\mathbf{B}^2$.

Product bilattices

Let $\mathbf{L} = \langle L; \sqcap, \sqcup \rangle$ be a lattice. The operations of the product bilattice

$$\mathbf{L} \odot \mathbf{L} = \langle L \times L; \otimes, \oplus, \wedge, \vee, \neg \rangle$$

are defined for $(a, b), (c, d) \in L \times L$ by

$$\begin{aligned}
(a, b) \otimes (c, d) &= (a \sqcap c, b \sqcap d) \\
(a, b) \oplus (c, d) &= (a \sqcup c, b \sqcup d) \\
(a, b) \wedge (c, d) &= (a \sqcap c, b \sqcup d) \\
(a, b) \vee (c, d) &= (a \sqcup c, b \sqcap d) \\
\neg(a, b) &= (b, a).
\end{aligned}$$

Think of an element $(a, b) \in L \times L$ as encoding evidence about some sentence: a is the evidence *for*, and b is the evidence *against*.

Product representation

The following theorem was proven by various researchers at different levels of generality. See Davey (2013) for a full historical account.

Theorem

A bilattice $\mathbf{B}$ *is interlaced if and only if it is isomorphic to the bilattice product* $\mathbf{L} \odot \mathbf{L}$ *for some lattice* $\mathbf{L}$.

This result can also be restricted to distributive bilattices.

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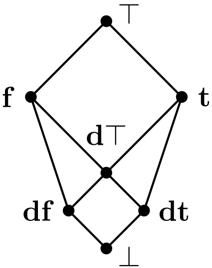
Ginsberg proposed the bilattice $\mathcal{SEVEN}$ for inference in default logic. The additional truth values $\mathbf{dt}$ and $\mathbf{df}$ represent *"true by default"* and *"false by default"*.



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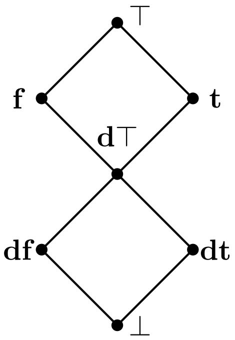
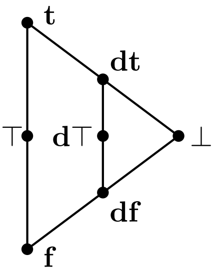
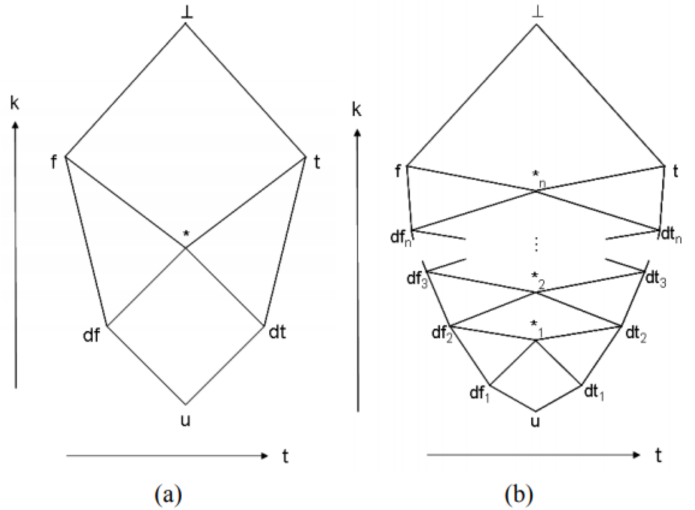




$\leqslant_k$ $\mathcal{SEVEN}$ $\leqslant_t$

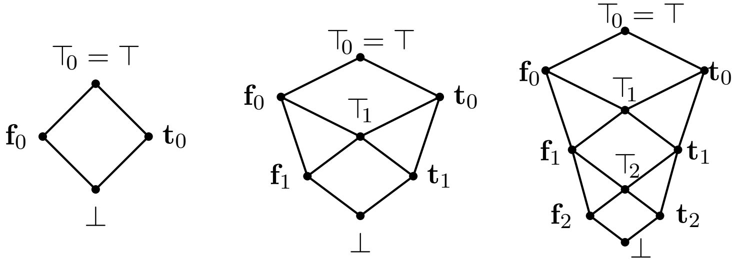
Note: $\mathcal{SEVEN}$ is not interlaced, as $\mathbf{d}\top \leqslant_k \mathbf{t}$ but
$\mathbf{d}\top \wedge \bot = \mathbf{df} \nleqslant_k \bot = \mathbf{t} \wedge \bot$.

Figure from Shet, Harwood, Davis (2006) "Multivalued default logic for identity maintenance in visual surveillance".



(a)Bilattice for default logic (b) Bilattice for prioritized default logic.

Consider a class of default bilattices which have a hierarchy of default truth values:

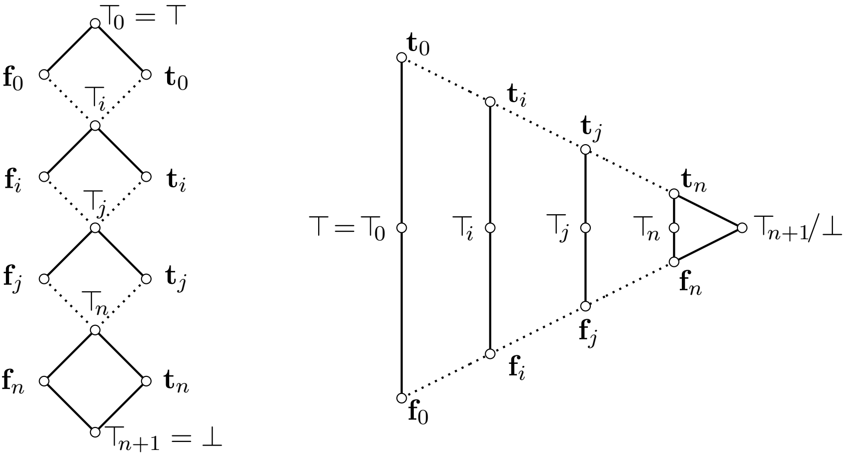


$\mathbf{K}_0$ $\mathbf{K}_1$ $\mathbf{K}_2$

For $n \in \omega$, we define $\mathbf{K}_n = \langle K_n; \otimes, \oplus, \wedge, \vee, \neg, \bot, \top \rangle$.

Note that $\mathbf{K}_n$ has no proper subalgebras.

A hierarchy of default bilattices Cabrer, C., Priestley (2015)



$\mathbf{K}_n$ in its knowledge order (left) and truth order (right); here $0 < i < j < n$.

Natural duality

Natural duality theory provides a uniform method for obtaining dual structures for algebras in a finitely generated quasivariety.

The theory has been developed by Clark, Davey, Haviar, Pitkethly, Priestley, Werner, Willard, and others. The main reference is the book "Natural duality for the working algebraist" by Clark and Davey (1998).

The theory has been successfully applied to many (quasi)varieties related to logic such as: distributive lattices (with additional operations), de Morgan algebras, Kleene algebras, finitely-generated quasivarieties of Heyting algebras.

Natural duality theory: broad outline

We are interested in a (finitely generated) quasivariety $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$ where $\mathbf{M}$ is a finite algebra.

If we can find a "suitable" structure to put on the underlying set of M, then for any $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ this structure can be transferred to the hom-set $\boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})$ to give us the **dual space** of $\mathbf{A}$. This suitable structure will include the discrete topology. We denote the structured and topologised version of $\mathbf{M}$ as $\underset{\sim}{\mathbf{M}}$.

Once a suitable structure has been found, the algebra $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ will be represented as the structure-preserving continuous maps from $\boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})$ to $\underset{\sim}{\mathbf{M}}$.

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Theorem (Clark and Davey, 1998)

(NU Duality Theorem, special case) *Let* $\mathbf{M}$ *be a finite lattice-based algebra and* $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$. *Let*

$$\underset{\sim}{\mathbf{M}} = \big(M; \mathbb{S}(\mathbf{M}^2), \mathcal{T} \big)$$

where $\mathcal{T}$ *is the discrete topology on* M. *Then* $\underset{\sim}{\mathbf{M}}$ *yields a duality on* $\boldsymbol{\mathcal{A}}$.

Note that the lattice $\mathbb{S}(\mathbf{M}^2)$ is an algebraic lattice and hence is meet-generated by its completely meet-irreducible elements (where meet is $\cap$).

Two further results from Natural Duality theory:

(i) If two relations R_1 and R_2 are in the dualising structure, then $R_1 \cap R_2$ is not required.

(ii) If R is in the dualising structure, then the converse relation is not required.

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Examples

Boolean algebras:

$\mathcal{B} = \mathbb{ISP}(\mathbf{M})$ where $\mathbf{M} = \langle \{0,1\}; \wedge, \vee, \neg, 0, 1 \rangle$

$\underset{\sim}{\mathbf{M}} = \langle \{0,1\}; \mathcal{T} \rangle$ and $\boldsymbol{\mathcal{X}} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}) =$ Stone spaces

Bounded distributive lattices:

$\mathcal{D} = \mathbb{ISP}(\mathbf{M})$ where $\mathbf{M} = \langle \{0,1\}; \wedge, \vee, 0, 1 \rangle$

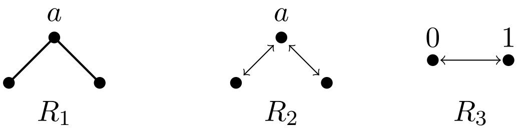
$\underset{\sim}{\mathbf{M}} = \langle \{0,1\}; \leqslant, \mathcal{T} \rangle$ and $\boldsymbol{\mathcal{X}} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}) =$ Priestley spaces

More examples

Kleene algebras:

$\boldsymbol{\mathcal{K}} = \mathbb{ISP}(\mathbf{K})$ where $\mathbf{K} = \langle \{0, a, 1\}; \wedge, \vee, \neg, 0, 1 \rangle$

$\underset{\sim}{\mathbf{K}} = \langle \{0, a, 1\}; R_1, R_2, R_3, \mathcal{T} \rangle$



Distributive bilattices:

$\boldsymbol{\mathcal{DB}} = \mathbb{ISP}(\mathcal{FOUR})$ where
$\mathcal{FOUR} = \langle \{\mathbf{f}, \mathbf{t}, \top, \bot\}; \otimes, \oplus, \wedge, \vee, \neg, \mathbf{f}, \mathbf{t} \rangle$

$\underset{\sim}{\mathcal{FOUR}} = \langle \{\mathbf{f}, \mathbf{t}, \top, \bot\}; \leqslant_k, \mathcal{T} \rangle$

$\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathcal{FOUR}}) =$ Priestley spaces

Now consider the default bilattices $\mathbf{K}_n$ $(n \in \omega)$ described earlier.

Proposition

The lattice $\mathbb{S}(\mathbf{K}_n^2)$ is isomorphic to the k-lattice reduct of $\mathbf{K}_n$.

Theorem (Cabrer, C., Priestley, 2015)

Consider $\mathcal{A} = \mathbb{ISP}(\mathbf{K}_n)$. Then $\underset{\sim}{\mathbf{K_n}} = (K_n; R_0, \dots, R_n, \mathcal{T})$ yields a strong (and hence full) duality on $\mathcal{A}$. Moreover, this duality is optimal.

For $a, b \in K_n$, we have $a <_k b$ if and only there exists R_i $(0 \leqslant i \leqslant n)$ such that $(a, b) \in R_i$ and $(b, a) \notin R_i$.

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For $a, b \in K_n$, we have $a <_k b$ if and only there exists R_i $(0 \leqslant i \leqslant n)$ such that $(a, b) \in R_i$ and $(b, a) \notin R_i$.

We say that $\preccurlyeq$ is a **Priestley quasi-order** if $x \not\preccurlyeq y$ implies that there exists a clopen set U that is a $\preccurlyeq$-up-set for which $x \in U$ and $y \notin U$.

Theorem (Cabrer, C., Priestley, 2015)

Let $\mathbf{X} = (X; \preccurlyeq_0, \preccurlyeq_1, \ldots, \preccurlyeq_n, \mathcal{T})$ be a structured topological space. Then $\mathbf{X} \in \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{K_n}})$ if and only if

(i) *$(X; \preccurlyeq_0, \mathcal{T})$ is a Priestley space;*

(ii) *$\preccurlyeq_i$ is a Priestley quasi-order extending $\preccurlyeq_0$, for $i \in \{1, \ldots, n\}$;*

(iii) *$\preccurlyeq_0 \subseteq \preccurlyeq_1 \subseteq \cdots \subseteq \preccurlyeq_n$;*

(iv) *$\preccurlyeq_i \subseteq \preccurlyeq_j$, for $0 \leqslant i < j \leqslant n$.*

Current and future work

We introduce a new family of default bilattices.

The bilattices $\mathbf{J}_0, \mathbf{J}_1$ and $\mathbf{J}_2$, drawn in their knowledge order.

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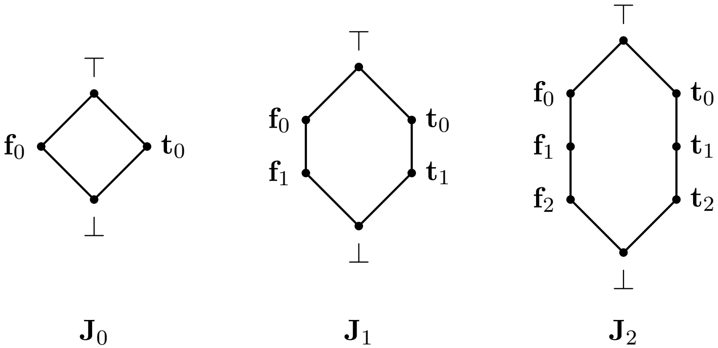
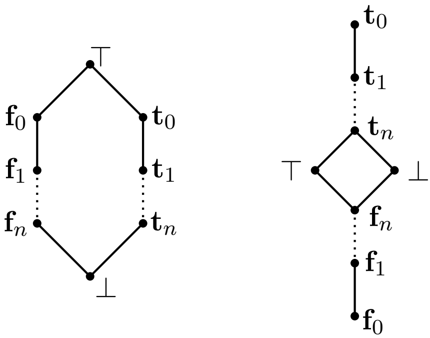


The bilattices $\mathbf{J}_0, \mathbf{J}_1$ and $\mathbf{J}_2$, drawn in their knowledge order.

The bilattice $\mathbf{J}_n$ drawn in both its knowledge (left) and truth (right) orders:



We let $\mathbf{J}_n = \langle J_n; \otimes, \oplus, \wedge, \vee, \neg, C \rangle$ where $C = J_n$ and $\neg \mathbf{t}_i = \mathbf{f}_i$ and $\neg \mathbf{f}_i = \mathbf{t}_i$.

Note: $\mathbf{J}_n$ is not interlaced for $n \geqslant 1$. We have $\mathbf{f}_0 \leqslant_k \top$ but

$$\bot \wedge \mathbf{f}_0 = \mathbf{f}_0 \nleqslant_k \mathbf{f}_n = \top \wedge \bot$$

We want to develop natural dualities for the quasivarieties $\mathbb{ISP}(\mathbf{J}_n)$ for $n \in \omega$.

Since the $\mathbf{J}_n$ are lattice-based, to find a duality we can study the lattice $\mathbb{S}(\mathbf{J}_n^2)$. We have

$$|\mathbb{S}(\mathbf{J}_0^2)| = 4 \quad \text{and} \quad |\mathbb{S}(\mathbf{J}_1^2)| = 7$$

but

$$|\mathbb{S}(\mathbf{J}_2^2)| = 28 \quad \text{and} \quad |\mathbb{S}(\mathbf{J}_3^2)| = 200$$

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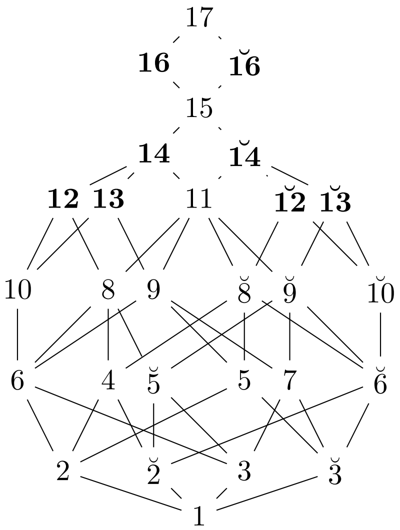

The algebraic lattice $\mathbb{S}(\mathbf{J}_2^2)$ with its completely meet-irreducible elements shown in bold.

For J_n and $i < n$, consider the binary relation

$$S_{n,i} := \{(\top,\top),(\bot,\bot)\} \cup \{\mathbf{f}_{i+1},\dots,\mathbf{f}_n\}^2 \cup \{\mathbf{t}_{i+1},\dots,\mathbf{t}_n\}^2$$
$$\cup \{(\mathbf{f}_j,\mathbf{f}_k) \mid i<j, 0 \leqslant k \leqslant i\} \cup \{(\mathbf{t}_j,\mathbf{t}_k) \mid i<j, 0 \leqslant k \leqslant i\}.$$

The binary relations $S_{3,0}, S_{3,1}$ and $S_{3,2}$ on J_3 drawn as quasi-orders. We draw x and y in the same block if xRy and yRx.

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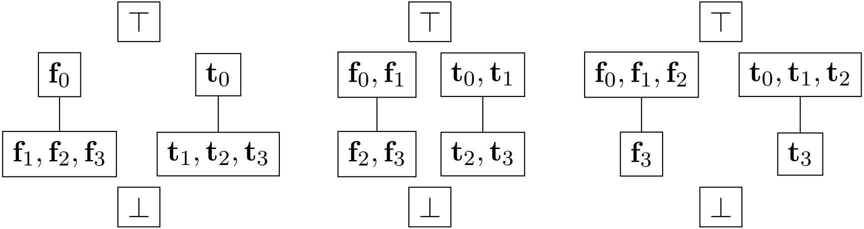

$\mathbf{S}_{3,0}$ $\mathbf{S}_{3,1}$ $\mathbf{S}_{3,2}$

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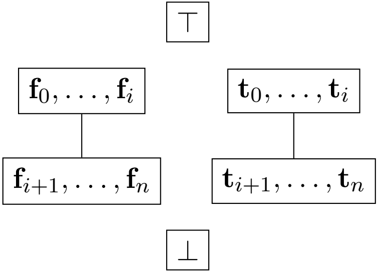
Proposition

Consider the bilattice $\mathbf{J}_n$. *For* $i \in \{0, 1, \ldots, n-1\}$ *let* $S_{n,i} \subseteq J_n^2$ *be defined by*

$$S_{n,i} := \{(\top,\top),(\bot,\bot)\} \cup \{\mathbf{f}_{i+1},\ldots,\mathbf{f}_n\}^2 \cup \{\mathbf{t}_{i+1},\ldots,\mathbf{t}_n\}^2$$
$$\cup \{(\mathbf{f}_j,\mathbf{f}_k) \mid i<j, 0 \leqslant k \leqslant i\} \cup \{(\mathbf{t}_j,\mathbf{t}_k) \mid i<j, 0 \leqslant k \leqslant i\}.$$

Then $S_{n,i}$ *is the universe of a subalgebra of* $\mathbf{J}_n^2$.



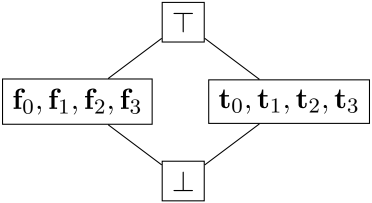
The binary algebraic relation $S_{n,i}$ drawn as a quasi-order

Proposition

Define the binary relation $S_{n,n}$ *on* J_n *by:*

$$S_{n,n} := \big(\{\bot\} \times J_n\big) \cup \big(J_n \times \{\top\}\big) \cup \{\mathbf{t}_0, \ldots, \mathbf{t}_n\}^2 \cup \{\mathbf{f}_0, \ldots, \mathbf{f}_n\}^2$$

Then $S_{n,n}$ *is the universe of a subalgebra of* $\mathbf{J}_n^2$.



$\mathbf{S}_{3,3}$

Theorem

The structure $\underset{\sim}{\mathbf{J}}_n := \langle J_n; S_{n,0}, \ldots, S_{n,n}, \mathcal{T} \rangle$ *yields a duality on the quasivariety* $\mathbb{ISP}(\mathbf{J}_n)$.