Primal and functionally complete algebras of logics

Radomír Halaš

coauthors: I. Chajda, I.G. Rosenberg

Department of Algebra and Geometry
Palacký University Olomouc
and
Department of Mathem. and Statistics
Université de Montréal

Banská Bystrica, April 15, 2014

Outline

Outline

Introduction

- classical logic: every boolean function $f : \{0,1\}^n \to \{0,1\}$ can be represented as a conjunction of elementary disjunctions (and vice versa)
- used in switching cirquits and logical design
- $\mathbf{A} = (A; F)$ is **primal** if every function f on A is a term function of $\mathbf{A}$
  $c_a : A \to A$ a constant unary function on A with value a
- A finite $\mathbf{A} = (A; F)$ is **functionally complete** if $(A; F \cup \{c_a\}_{a \in A})$ is primal

- classical logic: every boolean function $f : \{0,1\}^n \to \{0,1\}$ can be represented as a conjunction of elementary disjunctions (and vice versa)
- used in switching cirquits and logical design
- $\mathbf{A} = (A; F)$ is **primal** if every function f on A is a term function of $\mathbf{A}$
  $c_a : A \to A$ a constant unary function on A with value a
- A finite $\mathbf{A} = (A; F)$ is **functionally complete** if $(A; F \cup \{c_a\}_{a \in A})$ is primal

- Werner: A finite algebra is FC iff the ternary discriminator is its polynomial.
- 2-element BA: $t(x,y,z) = [(x \oplus y) \wedge x] \vee [(x \oplus y \oplus 1) \wedge z]$
- **weak logic**: $\mathbf{A} = (A; \{\to\} \cup F, 0, 1)$ with
  $1 \to x = x$, $0 \to x = 1$, $x \to 1 = 1$
  $x \to y = 1$ and $y \to x = 1$ iff $x = y$
- **weak conjunction**:
  $1 \odot x = x \odot 1 = x$
  $x \odot y = 1$ implies $x = y = 1$
- **strict negation**:
  $\neg 1 = 0$ and $\neg x = 1$ for $x \neq 1$
- **globalization**:
  $\Delta 1 = 1$ and $\Delta x = 0$ for $x \neq 1$
  for a strict negation $\neg$, $\Delta x = \neg\neg x$ is a globalization

- Werner: A finite algebra is FC iff the ternary discriminator is its polynomial.
- 2-element BA: $t(x,y,z) = [(x \oplus y) \wedge x] \vee [(x \oplus y \oplus 1) \wedge z]$
- **weak logic**: $\mathbf{A} = (A; \{\rightarrow\} \cup F, 0, 1)$ with
  $1 \rightarrow x = x$, $0 \rightarrow x = 1$, $x \rightarrow 1 = 1$
  $x \rightarrow y = 1$ and $y \rightarrow x = 1$ iff $x = y$
- **weak conjunction**:
  $1 \odot x = x \odot 1 = x$
  $x \odot y = 1$ implies $x = y = 1$
- **strict negation**:
  $\neg 1 = 0$ and $\neg x = 1$ for $x \neq 1$
- **globalization**:
  $\Delta 1 = 1$ and $\Delta x = 0$ for $x \neq 1$
  for a strict negation $\neg$, $\Delta x = \neg\neg x$ is a globalization

- Werner: A finite algebra is FC iff the ternary discriminator is its polynomial.
- 2-element BA: $t(x,y,z) = [(x \oplus y) \wedge x] \vee [(x \oplus y \oplus 1) \wedge z]$
- **weak logic**: $\mathbf{A} = (A; \{\rightarrow\} \cup F, 0, 1)$ with
  $1 \rightarrow x = x$, $0 \rightarrow x = 1$, $x \rightarrow 1 = 1$
  $x \rightarrow y = 1$ and $y \rightarrow x = 1$ iff $x = y$
- **weak conjunction**:
  $1 \odot x = x \odot 1 = x$
  $x \odot y = 1$ implies $x = y = 1$
- **strict negation**:
  $\neg 1 = 0$ and $\neg x = 1$ for $x \neq 1$
- **globalization**:
  $\Delta 1 = 1$ and $\Delta x = 0$ for $x \neq 1$
  for a strict negation $\neg$, $\Delta x = \neg\neg x$ is a globalization

Theorem

A weak logic with a weak conjunction and globalization is a discriminator algebra for

$$t(x,y,z) = [\Delta((x \to y) \odot (y \to x)) \to z] \odot [(\Delta((x \to y) \odot (y \to x)) \to 0) \to x].$$

Theorem

A finite weak logic with a weak conjunction is FC iff the globalization is its polynomial.

Corollary

Every finite chain considered as a relatively pseudocomplemented $\vee$-semilattice with the dual pseudocomplementation is FC.

Theorem

A weak logic with a weak conjunction and globalization is a discriminator algebra for

$$t(x,y,z) = [\Delta((x \to y) \odot (y \to x)) \to z] \odot [(\Delta((x \to y) \odot (y \to x)) \to 0) \to x].$$

Theorem

A finite weak logic with a weak conjunction is FC iff the globalization is its polynomial.

Corollary

Every finite chain considered as a relatively pseudocomplemented $\vee$-semilattice with the dual pseudocomplementation is FC.

Theorem

A weak logic with a weak conjunction and globalization is a discriminator algebra for

$$t(x,y,z) = [\Delta((x \to y) \odot (y \to x)) \to z] \odot [(\Delta((x \to y) \odot (y \to x)) \to 0) \to x].$$

Theorem

A finite weak logic with a weak conjunction is FC iff the globalization is its polynomial.

Corollary

Every finite chain considered as a relatively pseudocomplemented $\vee$*-semilattice with the dual pseudocomplementation is FC.*

Problem setting

- $A \ldots$ finite universe, $|A| > 1$, $0, 1$ two distinct fixed element of A, $f : A^2 \to A$ binary function satisfying for all $x \in A$:

$$f(0,x) = f(x,1) = f(x,x) = 1, \;\; f(1,x) = x. \tag{1}$$

We often write $x \to y$ instead of $f(x,y)$.

Problem: Let X be a set of operations on A such that $f, c_0 \in X$. When is the algebra $\langle A, X \rangle$ primal or equivalently X complete?

Problem setting

- $A \ldots$ finite universe, $|A| > 1$, $0, 1$ two distinct fixed element of A, $f : A^2 \to A$ binary function satisfying for all $x \in A$:

$$f(0,x) = f(x,1) = f(x,x) = 1, \ f(1,x) = x. \tag{1}$$

We often write $x \to y$ instead of $f(x,y)$.

Problem: Let X be a set of operations on A such that $f, c_0 \in X$. When is the algebra $\langle A, X \rangle$ primal or equivalently X complete?

Clones

- $\mathscr{O}$ ... the set of all finitary operations on A
- $\mathscr{C} \subseteq \mathscr{O}$ is a **clone** if $\mathscr{C}$ is a composition closed subset of $\mathscr{O}$ containing all n-ary projections e_i^n (where $e_i^n(x_1, \dots, x_n) = x_i$)
  $=$ the set of term operations on A
- $X \subseteq \mathscr{O}$ is complete (or $\langle A; X \rangle$ is primal) if $\mathscr{O}$ is the only clone containing X
- ρ ... h-ary relation on A ($=$ a set of h-tuples over A)
  g ... n-ary operation on A
  g **preserves** ρ if for every $h \times n$ matrix with row vectors $\mathbf{r_1}, \dots, \mathbf{r_h}$ and column vectors $\mathbf{c_1}, \dots, \mathbf{c_n} \in \rho$

$$(g(\mathbf{r_1}), \dots, g(\mathbf{r_h})) \in \rho.$$

  $\Leftrightarrow$ ρ is a subuniverse of $\langle A; g \rangle^h$.
- Polρ ... the set of operations on A preserving ρ

Clones

- $\mathscr{O} \ldots$ the set of all finitary operations on A
- $\mathscr{C} \subseteq \mathscr{O}$ is a **clone** if $\mathscr{C}$ is a composition closed subset of $\mathscr{O}$ containing all n-ary projections e_i^n (where $e_i^n(x_1, \ldots, x_n) = x_i$)
  $=$ the set of term operations on A
- $X \subseteq \mathscr{O}$ is complete (or $\langle A; X \rangle$ is primal) if $\mathscr{O}$ is the only clone containing X
- $\rho \ldots h$-ary relation on A ($=$ a set of h-tuples over A)
  $g \ldots n$-ary operation on A
  g **preserves** ρ if for every $h \times n$ matrix with row vectors $\mathbf{r_1}, \ldots, \mathbf{r_h}$ and column vectors $\mathbf{c_1}, \ldots, \mathbf{c_n} \in \rho$

$$(g(\mathbf{r_1}), \ldots, g(\mathbf{r_h})) \in \rho.$$

  $\Leftrightarrow \rho$ is a subuniverse of $\langle A; g \rangle^h$.
- Pol$\rho \ldots$ the set of operations on A preserving ρ

Clones

- $\mathscr{O}\ldots$ the set of all finitary operations on A
- $\mathscr{C} \subseteq \mathscr{O}$ is a **clone** if $\mathscr{C}$ is a composition closed subset of $\mathscr{O}$ containing all n-ary projections e_i^n (where $e_i^n(x_1,\ldots,x_n) = x_i$)
  $=$ the set of term operations on A
- $X \subseteq \mathscr{O}$ is complete (or $\langle A; X\rangle$ is primal) if $\mathscr{O}$ is the only clone containing X
- ρ ... h-ary relation on A ($=$ a set of h-tuples over A)
  g ... n-ary operation on A
  g **preserves** ρ if for every $h \times n$ matrix with row vectors $\mathbf{r_1},\ldots,\mathbf{r_h}$ and column vectors $\mathbf{c_1},\ldots,\mathbf{c_n} \in \rho$

$$(g(\mathbf{r_1}),\ldots,g(\mathbf{r_h})) \in \rho.$$

  $\Leftrightarrow \rho$ is a subuniverse of $\langle A; g\rangle^h$.
- Polρ ... the set of operations on A preserving ρ

Clones

- $\mathscr{O}$ . . . the set of all finitary operations on A
- $\mathscr{C} \subseteq \mathscr{O}$ is a **clone** if $\mathscr{C}$ is a composition closed subset of $\mathscr{O}$ containing all n-ary projections e_i^n (where $e_i^n(x_1, \ldots, x_n) = x_i$)
  $=$ the set of term operations on A
- $X \subseteq \mathscr{O}$ is complete (or $\langle A; X \rangle$ is primal) if $\mathscr{O}$ is the only clone containing X
- ρ . . . h-ary relation on A ($=$ a set of h-tuples over A)
  g . . . n-ary operation on A
  g **preserves** ρ if for every $h \times n$ matrix with row vectors $\mathbf{r_1}, \ldots, \mathbf{r_h}$ and column vectors $\mathbf{c_1}, \ldots, \mathbf{c_n} \in \rho$

$$(g(\mathbf{r_1}), \ldots, g(\mathbf{r_h})) \in \rho.$$

  $\Leftrightarrow$ ρ is a subuniverse of $\langle A; g \rangle^h$.
- Polρ . . . the set of operations on A preserving ρ

To formulate our main result we need the following three types of relations.

1. A **unary** relation on A is subset of A.

2. Let $h \geq 2$ and ρ an h-ary relation on A. The relation ρ is **totally reflexive** if $(a_1, \ldots, a_h) \in \rho$ whenever $a_1, \ldots, a_h \in A$ are not pairwise distinct and ρ is **totally symmetric** if for every permutation π of $\{1, \ldots, h\}$

$$(a_1, \ldots, a_h) \in \rho \quad \Rightarrow \quad (a_{\pi(1)}, \ldots, a_{\pi(h)}) \in \rho.$$

An element $c \in A$ is **central** for a totally reflexive and symmetric h-ary relation ρ if $(c, a_2, \ldots, a_h) \in \rho$ for all $a_2, \ldots, a_h \in A$. The **center** C_ρ of ρ is the set of central elements of ρ. A totally reflexive and symmetric relation ρ is **central** if $\emptyset \neq C_\rho \subset A$. Notice that for $h = 1$ every unary relation is trivially totally reflexive and totally symmetric.

To formulate our main result we need the following three types of relations.

1. A **unary** relation on A is subset of A.

2. Let $h \geq 2$ and ρ an h-ary relation on A. The relation ρ is **totally reflexive** if $(a_1, \ldots, a_h) \in \rho$ whenever $a_1, \ldots, a_h \in A$ are not pairwise distinct and ρ is **totally symmetric** if for every permutation π of $\{1, \ldots, h\}$

$$(a_1, \ldots, a_h) \in \rho \quad \Rightarrow \quad (a_{\pi(1)}, \ldots, a_{\pi(h)}) \in \rho.$$

An element $c \in A$ is **central** for a totally reflexive and symmetric h-ary relation ρ if $(c, a_2, \ldots, a_h) \in \rho$ for all $a_2, \ldots, a_h \in A$. The **center** C_ρ of ρ is the set of central elements of ρ. A totally reflexive and symmetric relation ρ is **central** if $\emptyset \neq C_\rho \subset A$. Notice that for $h = 1$ every unary relation is trivially totally reflexive and totally symmetric.

To formulate our main result we need the following three types of relations.

1. A **unary** relation on A is subset of A.

2. Let $h \geq 2$ and ρ an h-ary relation on A. The relation ρ is **totally reflexive** if $(a_1, \ldots, a_h) \in \rho$ whenever $a_1, \ldots, a_h \in A$ are not pairwise distinct and ρ is **totally symmetric** if for every permutation π of $\{1, \ldots, h\}$

$$(a_1, \ldots, a_h) \in \rho \quad \Rightarrow \quad (a_{\pi(1)}, \ldots, a_{\pi(h)}) \in \rho.$$

An element $c \in A$ is **central** for a totally reflexive and symmetric h-ary relation ρ if $(c, a_2, \ldots, a_h) \in \rho$ for all $a_2, \ldots, a_h \in A$. The **center** C_ρ of ρ is the set of central elements of ρ. A totally reflexive and symmetric relation ρ is **central** if $\emptyset \neq C_\rho \subset A$.
Notice that for $h = 1$ every unary relation is trivially totally reflexive and totally symmetric.

To formulate our main result we need the following three types of relations.

1. A **unary** relation on A is subset of A.

2. Let $h \geq 2$ and ρ an h-ary relation on A. The relation ρ is **totally reflexive** if $(a_1, \ldots, a_h) \in \rho$ whenever $a_1, \ldots, a_h \in A$ are not pairwise distinct and ρ is **totally symmetric** if for every permutation π of $\{1, \ldots, h\}$

$$(a_1, \ldots, a_h) \in \rho \quad \Rightarrow \quad (a_{\pi(1)}, \ldots, a_{\pi(h)}) \in \rho.$$

An element $c \in A$ is **central** for a totally reflexive and symmetric h-ary relation ρ if $(c, a_2, \ldots, a_h) \in \rho$ for all $a_2, \ldots, a_h \in A$. The **center** C_ρ of ρ is the set of central elements of ρ. A totally reflexive and symmetric relation ρ is **central** if $\emptyset \neq C_\rho \subset A$. Notice that for $h = 1$ every unary relation is trivially totally reflexive and totally symmetric.

Clones

If ρ is an equivalence relation on A and $a \in A$ we denote by $[a]_\rho$ the block of ρ containing a.

A clone $\mathscr{C} \neq \mathscr{O}$ is **maximal** (or precomplete) if $\mathscr{O}$ is the only clone properly containing $\mathscr{C}$.

(i) every clone $\mathscr{C} \neq \mathscr{O}$ is contained in a maximal clone

(ii) every maximal clone is of the form $\mathrm{Pol}\rho$ where ρ belongs to one of six sets of relations on A desribed in I – VI below.

Moreover, $X \subseteq A$ **is complete if and only if** X **is included in no maximal clone.**

If ρ is an equivalence relation on A and $a \in A$ we denote by $[a]_\rho$ the block of ρ containing a.
A clone $\mathscr{C} \neq \mathscr{O}$ is **maximal** (or precomplete) if $\mathscr{O}$ is the only clone properly containing $\mathscr{C}$.

(i) every clone $\mathscr{C} \neq \mathscr{O}$ is contained in a maximal clone

(ii) every maximal clone is of the form $\mathrm{Pol}\rho$ where ρ belongs to one of six sets of relations on A desribed in I – VI below.

Moreover, $X \subseteq A$ **is complete if and only if** X **is included in no maximal clone.**

If ρ is an equivalence relation on A and $a \in A$ we denote by $[a]_\rho$ the block of ρ containing a.
A clone $\mathscr{C} \neq \mathscr{O}$ is **maximal** (or precomplete) if $\mathscr{O}$ is the only clone properly containing $\mathscr{C}$.

(i) every clone $\mathscr{C} \neq \mathscr{O}$ is contained in a maximal clone

(ii) every maximal clone is of the form $\mathrm{Pol}\rho$ where ρ belongs to one of six sets of relations on A desribed in I – VI below.

Moreover, $X \subseteq A$ **is complete if and only if** X **is included in no maximal clone.**

If ρ is an equivalence relation on A and $a \in A$ we denote by $[a]_\rho$ the block of ρ containing a.

A clone $\mathscr{C} \neq \mathscr{O}$ is **maximal** (or precomplete) if $\mathscr{O}$ is the only clone properly containing $\mathscr{C}$.

(i) every clone $\mathscr{C} \neq \mathscr{O}$ is contained in a maximal clone

(ii) every maximal clone is of the form $\mathrm{Pol}\rho$ where ρ belongs to one of six sets of relations on A desribed in I – VI below.

Moreover, $X \subseteq A$ **is complete if and only if** X **is included in no maximal clone.**

If ρ is an equivalence relation on A and $a \in A$ we denote by $[a]_\rho$ the block of ρ containing a.
A clone $\mathscr{C} \neq \mathscr{O}$ is **maximal** (or precomplete) if $\mathscr{O}$ is the only clone properly containing $\mathscr{C}$.

(i) every clone $\mathscr{C} \neq \mathscr{O}$ is contained in a maximal clone

(ii) every maximal clone is of the form $\mathrm{Pol}\rho$ where ρ belongs to one of six sets of relations on A desribed in I – VI below.

Moreover, $X \subseteq A$ **is complete if and only if** X **is included in no maximal clone.**

Theorem

Let A be a finite set of cardinality at least 3, let $0,1 \in A$, let f be a binary operation on A satisfying (1) and let X be a set of operations on A containing c_0 and f. Then X is complete if and only if $X \setminus \mathrm{Pol}\rho$ is nonvoid for each of the following relations:

(i) ρ *is unary,* $0,1 \in \rho \subset A$ *and*
$\{f(x,y) : x \in \rho \setminus \{0,1\}, y \in \rho \setminus \{x,1\}\} \subseteq \rho$;

(ii) ρ *is an equivalence relation on A with at least two blocks and at least one nonsingleton block such that*

  a) $f(([0] \times A) \cup (A \times [1]) \cup \bigcup_{a \in A \setminus \{0,1\}}([a] \times [a])) \subseteq [1]$,

  b) $f([1] \times [a]) \subseteq [a]$ *for all* $a \in A$, *and*

  c) *the set* $f([a] \times [b])$ *is included in a block of* ρ *provided* $[0] \neq [a] \neq [b] \neq [1] \neq [a]$;

(iii) $h \geq 2$, ρ *is an h-ary central relation with* $1 \in C_\rho$ *satisfying for all* $a_1, \ldots, a_h, c \in A$

$$(a_1, \ldots, a_h) \in \rho \quad \Rightarrow \quad (f(c, a_1), a_2, \ldots, a_h) \in \rho.$$

Proof: based on the Rosenberg's characterization of maximal clones of a given type containg c_0 and f.
Types of maximal clones:
I. $h = 1$, $\emptyset \neq \rho \subset A$

II. π is a permutation of A consisting of cycles of prime lenght p where p divides the cardinality of A,

$$\pi^{\diamond} = \{(x, \pi(x)) : x \in A\}$$

III. The cardinality of A is p^m, p is a prime.
Identify A with $\mathbf{p}^m$, the set of all m-tuples over $\mathbf{p} = \{0, \ldots, p-1\}$.
Denote by $\oplus$ the mod p sum on $\mathbf{p}$.
For $x = (x_1, \ldots, x_m) \in \mathbf{p}^m$ and $y = (y_1, \ldots, y_m) \in \mathbf{p}^m$ set
$x + y = (x_1 \oplus y_1, \ldots, x_m \oplus y_m)$.
Then $(\mathbf{p}^m, +)$ is an abelian p-group.
Denote by ρ the quaternary relation on $\mathbf{p}^m$

$$\{(a, b, c, a+b+c) : a, b, c \in \mathbf{p}^m\}.$$

Proof: based on the Rosenberg's characterization of maximal clones of a given type containg c_0 and f.
Types of maximal clones:
I. $h = 1$, $\emptyset \neq \rho \subset A$

II. π is a permutation of A consisting of cycles of prime lenght p where p divides the cardinality of A,

$$\pi^\diamond = \{(x, \pi(x)) : x \in A\}$$

III. The cardinality of A is p^m, p is a prime.
Identify A with $\mathbf{p}^m$, the set of all m-tuples over $\mathbf{p} = \{0, \ldots, p-1\}$.
Denote by $\oplus$ the mod p sum on $\mathbf{p}$.
For $x = (x_1, \ldots, x_m) \in \mathbf{p}^m$ and $y = (y_1, \ldots, y_m) \in \mathbf{p}^m$ set
$x + y = (x_1 \oplus y_1, \ldots, x_m \oplus y_m)$.
Then $(\mathbf{p}^m, +)$ is an abelian p-group.
Denote by ρ the quaternary relation on $\mathbf{p}^m$

$$\{(a, b, c, a + b + c) : a, b, c \in \mathbf{p}^m\}.$$

Proof: based on the Rosenberg's characterization of maximal clones of a given type containg c_0 and f.
Types of maximal clones:
I. $h = 1$, $\emptyset \neq \rho \subset A$

II. π is a permutation of A consisting of cycles of prime lenght p where p divides the cardinality of A,

$$\pi^\diamond = \{(x, \pi(x)) : x \in A\}$$

III. The cardinality of A is p^m, p is a prime.
Identify A with $\mathbf{p}^m$, the set of all m-tuples over $\mathbf{p} = \{0, \ldots, p-1\}$.
Denote by $\oplus$ the mod p sum on $\mathbf{p}$.
For $x = (x_1, \ldots, x_m) \in \mathbf{p}^m$ and $y = (y_1, \ldots, y_m) \in \mathbf{p}^m$ set $x + y = (x_1 \oplus y_1, \ldots, x_m \oplus y_m)$.
Then $(\mathbf{p}^m, +)$ is an abelian p-group.
Denote by ρ the quaternary relation on $\mathbf{p}^m$

$$\{(a, b, c, a+b+c) : a, b, c \in \mathbf{p}^m\}.$$

Proof: based on the Rosenberg's characterization of maximal clones of a given type containg c_0 and f.
Types of maximal clones:
I. $h = 1$, $\emptyset \neq \rho \subset A$

II. π is a permutation of A consisting of cycles of prime lenght p where p divides the cardinality of A,

$$\pi^\diamond = \{(x, \pi(x)) : x \in A\}$$

III. The cardinality of A is p^m, p is a prime.
Identify A with $\mathbf{p}^m$, the set of all m-tuples over $\mathbf{p} = \{0, \ldots, p-1\}$.
Denote by $\oplus$ the mod p sum on $\mathbf{p}$.
For $x = (x_1, \ldots, x_m) \in \mathbf{p}^m$ and $y = (y_1, \ldots, y_m) \in \mathbf{p}^m$ set
$x + y = (x_1 \oplus y_1, \ldots, x_m \oplus y_m)$.
Then $(\mathbf{p}^m, +)$ is an abelian p-group.
Denote by ρ the quaternary relation on $\mathbf{p}^m$

$$\{(a, b, c, a+b+c) : a, b, c \in \mathbf{p}^m\}.$$

IV. $\leq$ is a bounded order on A

V. ρ is an equivalence relation on A with at least two blocks and a nonsingleton block

VI. $h \geq 2$ and ρ is an h-ary central relation on A.

IV. $\leq$ is a bounded order on A

V. ρ is an equivalence relation on A with at least two blocks and a nonsingleton block

VI. $h \geq 2$ and ρ is an h-ary central relation on A.

IV. $\leq$ is a bounded order on A

V. ρ is an equivalence relation on A with at least two blocks and a nonsingleton block

VI. $h \geq 2$ and ρ is an h-ary central relation on A.

Outline

- every algebra of logic contains at least an operation implication (denoted by $\to$)
- the presence of implication alone is not sufficient to prove that $(A; \to)$ is functionally complete – we need another logical operations to exclude the remaining cases in Theorem.
- a binary operation $\odot$ on A is called a **weak conjunction** if it satisfies $x \odot 1 = 1 \odot x = x$
- a unary operation $\neg$ on A is called a **strict negation** if $\neg 1 = 0$ and $\neg x = 1$ otherwise
- consider a one more condition:

$$(x \to y = 1 \text{ and } y \to x = 1) \text{ implies } x = y. \qquad \text{(AS)}$$

- every algebra of logic contains at least an operation implication (denoted by $\to$)
- the presence of implication alone is not sufficient to prove that $(A; \to)$ is functionally complete – we need another logical operations to exclude the remaining cases in Theorem.
- a binary operation $\odot$ on A is called a **weak conjunction** if it satisfies $x \odot 1 = 1 \odot x = x$
- a unary operation $\neg$ on A is called a **strict negation** if $\neg 1 = 0$ and $\neg x = 1$ otherwise
- consider a one more condition:

$$(x \to y = 1 \text{ and } y \to x = 1) \text{ implies } x = y. \qquad \text{(AS)}$$

Applications

- every algebra of logic contains at least an operation implication (denoted by $\to$)
- the presence of implication alone is not sufficient to prove that $(A; \to)$ is functionally complete – we need another logical operations to exclude the remaining cases in Theorem.
- a binary operation $\odot$ on A is called a **weak conjunction** if it satisfies $x \odot 1 = 1 \odot x = x$
- a unary operation $\neg$ on A is called a **strict negation** if $\neg 1 = 0$ and $\neg x = 1$ otherwise
- consider a one more condition:

$$(x \to y = 1 \text{ and } y \to x = 1) \text{ implies } x = y. \qquad \text{(AS)}$$

Lemma

Let $(A;X)$ be a finite algebra and $\rightarrow$ satisfies (AS). Let $\neg$ be a strict negation and $\rightarrow, \neg \in X$. Then $(A;X)$ is functionally complete if and only if $X \setminus \mathrm{Pol}\rho \neq \emptyset$ for every h-ary central relation ρ satisfying (iii) of Theorem.

Lemma

Let $(A;X)$ be a finite algebra and $\rightarrow$ satisfies (AS). Let $\odot$ be a weak conjuction and $\rightarrow, \odot \in X$. Then $\odot \notin \mathrm{Pol}\rho$ for every h-ary central relation $(h \geq 2)$ $\rho \neq A^h$ with $1 \in C_\rho$.

Lemma

Let $(A;X)$ *be a finite algebra and* $\to$ *satisfies (AS). Let* $\neg$ *be a strict negation and* $\to,\neg \in X$. *Then* $(A;X)$ *is functionally complete if and only if* $X \setminus \mathrm{Pol}\rho \neq \emptyset$ *for every h-ary central relation* ρ *satisfying (iii) of Theorem.*

Lemma

Let $(A;X)$ *be a finite algebra and* $\to$ *satisfies (AS). Let* $\odot$ *be a weak conjuction and* $\to,\odot \in X$. *Then* $\odot \notin \mathrm{Pol}\rho$ *for every h-ary central relation* $(h \geq 2)$ $\rho \neq A^h$ *with* $1 \in C_\rho$.

Theorem

Let $(A; X)$ be a finite algebra such that $\rightarrow, \odot, \neg \in X$ where $\rightarrow$ satisfies (AS), $\odot$ is a weak conjuction and $\neg$ is a strict negation. Then $(A; X)$ is functionally complete.

- **pocrim** . . . partially ordered commutative residuated integral monoid, i.e. a structure $(A, \cdot, \to, 1)$ of type $(2, 2, 0)$ where
  (i) $(A, \leq, \cdot, 1)$ is a partially ordered commutative monoid where
  $$x \leq y \quad \text{iff} \quad x \to y = y \to x = 1$$
  (ii) $(A, \leq, \cdot)$ is residuated by $\to$, i.e. the following adjointness condition holds on A:
  $$z \leq x \to y \quad \text{iff} \quad z \cdot x \leq y.$$
- **bounded pocrim** . . . has a least element 0
- **residuated semilattices** and **residuated lattices** are special pocrims
- pocrims do not form a variety, but residuated semilattices form a variety

Pocrims

- **pocrim** . . . partially ordered commutative residuated integral monoid, i.e. a structure $(A, \cdot, \to, 1)$ of type $(2, 2, 0)$ where
  (i) $(A, \leq, \cdot, 1)$ is a partially ordered commutative monoid where
  $$x \leq y \quad \text{iff} \quad x \to y = y \to x = 1$$
  (ii) $(A, \leq, \cdot)$ is residuated by $\to$, i.e. the following adjointness condition holds on A:
  $$z \leq x \to y \quad \text{iff} \quad z \cdot x \leq y.$$
- **bounded pocrim** . . . has a least element 0
- **residuated semilattices** and **residuated lattices** are special pocrims
- pocrims do not form a variety, but residuated semilattices form a variety

- Given a pocrim $(A, \cdot, \rightarrow, 1)$, $F \subseteq A$ with $1 \in F$ is called a **filter** if
  (i) $a \cdot b \in F$ whenever $a, b \in F$
  (ii) if $a \leq b$ and $a \in F$ then $b \in F$.

  By a **deductive system** on a pocrim $(A, \cdot, \rightarrow, 1)$ we mean a subset $D \subseteq A$ with $1 \in D$ closed under **modus ponens**, i.e. if $a \rightarrow b \in D$ and $a \in D$ imply $b \in D$.
- A pocrim $\mathbf{A} = (A, \cdot, \rightarrow, 1)$ is said to be **H-closed** if $\mathbf{A}/\theta$ is a pocrim for all $\theta \in Con\mathbf{A}$.
- for H-closed pocrims we have

  deductive systems $\leftrightarrow$ filters $\leftrightarrow$ congruence kernels $\leftrightarrow$ congruences

Pocrims

- Given a pocrim $(A, \cdot, \to, 1)$, $F \subseteq A$ with $1 \in F$ is called a **filter** if
  (i) $a \cdot b \in F$ whenever $a, b \in F$
  (ii) if $a \leq b$ and $a \in F$ then $b \in F$.

  By a **deductive system** on a pocrim $(A, \cdot, \to, 1)$ we mean a subset $D \subseteq A$ with $1 \in D$ closed under **modus ponens**, i.e. if $a \to b \in D$ and $a \in D$ imply $b \in D$.
- A pocrim $\mathbf{A} = (A, \cdot, \to, 1)$ is said to be **H-closed** if $\mathbf{A}/\theta$ is a pocrim for all $\theta \in Con\mathbf{A}$.
- for *H*-closed pocrims we have

$$\text{deductive systems} \leftrightarrow \text{filters} \leftrightarrow \text{congruence kernels} \leftrightarrow \text{congruences}$$

- common fuzzy logical calculi are often too weak from the point of view of applications
- Hájek proposed the extension of BL-logic by adding a new unary logical connective ”very true” (vt-operator)
- vt-operators were successfully used in formal concept analysis (FCA)
  aim in FCA: to extract interesting clusters (formal concepts from tabular data)
  the number of formal concepts in data can be extremely large
  hedges - reducing the number of concepts

Pocrims – wvt-operators

- common fuzzy logical calculi are often too weak from the point of view of applications
- Hájek proposed the extension of BL-logic by adding a new unary logical connective ”very true” (vt-operator)
- vt-operators were successfully used in formal concept analysis (FCA)
  aim in FCA: to extract interesting clusters (formal concepts from tabular data)
  the number of formal concepts in data can be extremely large
  hedges - reducing the number of concepts

Definition

Let $\mathbf{A} = (A; \cdot, \to, 1)$ be a pocrim. A mapping ${}^\star : A \to A$ is called a **weak vt-operator** on $\mathbf{A}$ (wvt-operator in brief) if for any $x, y \in A$:

(1) $1^\star = 1$

(2) $x^\star \leq x$ (i.e., * is subdiagonal)

(3) $(x \to y)^\star \leq x^\star \to y^\star$.

If a wvt-operator * satisfies for any $x \in A$

(4) $x^\star = x^{\star\star}$ . . . **hedge**.

There are two boundary cases of hedges:

(i) identity, i.e. $a^\star = a$ for all $a \in A$

(ii) globalization: $1^\star = 1$ and $x^\star = 0$ otherwise

Definition

Let $\mathbf{A} = (A; \cdot, \rightarrow, 1)$ be a pocrim. A mapping $^\star : A \rightarrow A$ is called a **weak vt-operator** on $\mathbf{A}$ (wvt-operator in brief) if for any $x, y \in A$:

(1) $1^\star = 1$

(2) $x^\star \leq x$ (i.e., * is subdiagonal)

(3) $(x \rightarrow y)^\star \leq x^\star \rightarrow y^\star$.
If a wvt-operator * satisfies for any $x \in A$

(4) $x^\star = x^{\star\star}$ . . . **hedge.**

There are two boundary cases of hedges:

(i) identity, i.e. $a^\star = a$ for all $a \in A$

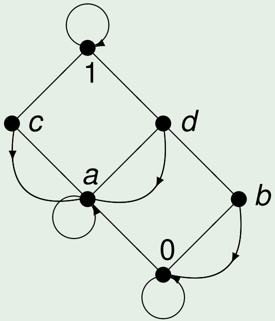
(ii) globalization: $1^\star = 1$ and $x^\star = 0$ otherwise

Theorem

A finite bounded H-closed pocrim $\mathbf{A} = (A, \cdot, \rightarrow, 0, 1, ^\star)$ *with a wvt-operator* * *is functionally complete iff* $\mathbf{A}$ *is simple.*

Clearly, if * is a globalization on A, then for $\theta \in Con(A, \cdot, \rightarrow, 0, 1)$ with $\theta \neq \omega$, * connects 0 and 1, and thus $(A, \cdot, \rightarrow, 0, 1, ^\star)$ is simple: indeed, $(0, 1) \in \theta$ yields $(1, x) = (0 \rightarrow x, 1 \rightarrow x) \in \theta$ for all $x \in A$ and hence $\theta = A^2$.

Example



- * visualized by arrows is hedge on $\mathscr{A}$ distinct from a globalization.
- Two non-trivial cogruences $\theta_1 = \{\{0, a, c\}, \{b, d, 1\}\}$ and $\theta_2 = \{\{0, b\}, \{c, 1\}, \{a, d\}\}$, both of them generate A^2 on $\mathbf{A} = (A; \cdot, \rightarrow, 0, 1, ^\star)$.

- If $a \in A$, let $F(a)$ denotes a filter in $(A; \cdot, \rightarrow, 0, 1)$ generated by a and let $\mathscr{F}(X)$ be a congruence kernel on $(A, \cdot, \rightarrow, 0, 1, ^\star)$ generated by $X \subseteq A$.
- Given a wvt-operator on A and $a \in A$, define inductively
$$a_1 := a^\star,\ a_{n+1} := a_n^\star.$$
Since A is finite and each wvt-operator * is subdiagonal, for every $a \in A$ there is a least $j_a \in \mathbb{N}$ for which $a_{j_a} = a_{j_a+1}$. Clearly, if * is a hedge, then $j_a = 1$ for all $a \in A$.
- $F \subseteq A$ is a congruence kernel on $(A; \cdot, \rightarrow, 0, 1, ^\star)$ iff *F is a filter on* $(A, \cdot, \rightarrow, 0, 1)$ which is *-closed, i.e. $a^\star \in F$ for all $a \in F$.
- $\mathscr{F}(F(a)) = F(a_{j_a})$.

Theorem

Let $\mathbf{A} = (A; \cdot, \to, 0, 1, ^{\star})$ *be a finite bounded H-closed pocrim with a wvt-operator* **. Then the following are equivalent:*

(i) $\mathbf{A}$ *is functionally complete*

(ii) *for every coatom* $a \in A$ *with* $F(a) \neq A$ *we have* $F(a_{j_a}) = A$*.*

Corollary

Let $\mathbf{A} = (A; \cdot, \to, 0, 1, ^{\star})$ *be a finite bounded H-closed pocrim with a hedge* **. Then the following are equivalent:*

(i) $\mathbf{A}$ *is functionally complete*

(ii) *for every coatom* $a \in A$ *with* $F(a) \neq A$ *we have* $F(a^{\star}) = A$*.*

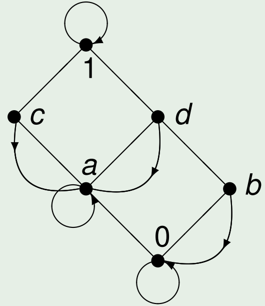
Corollary

Let $\mathbf{A} = (A; \cdot, \rightarrow, 0, 1, ^\star)$ *be a finite bounded residuated semilattice with a wvt-operator* **. Then the following are equivalent:*

(i) $\mathbf{A}$ *is functionally complete*

(ii) *for every coatom* $a \in A$ *with* $F(a) \neq A$ *we have* $F(a_{j_a}) = A$*.*

Example



There are two minimal non-trivial filters on $(A, \cdot, \to, 0, 1)$, namely $F(c) = \{1, c\}$ and $F(d) = \{1, b, d\}$. Further,

$$c = c_1 > a = c^\star = c_2 = a^\star = c_3,$$
$$d = d_1 > a = d^\star = d_2 = a^\star = d_3,$$

thus $j_c = j_d = 2$. We have $F(c_2) = F(d_2) = F(a) = A$, hence $(A, \cdot, \to, 0, 1, ^\star)$ is functionally complete.