One-dimensional continua and inverse limits

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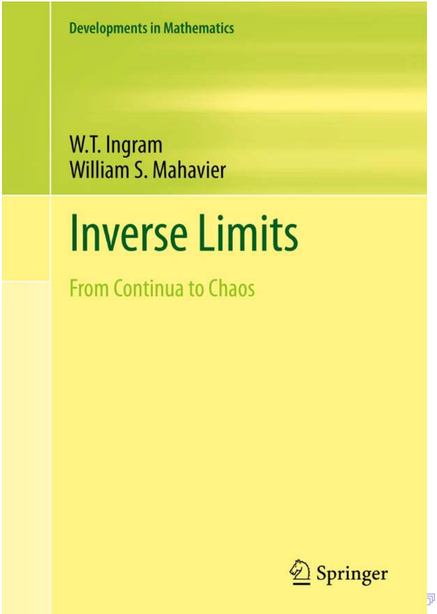

Contents

One-dimensional continua

X is a continuum:

- X is a compact metric space
- X is connected
  - A, B: open disjoint, $X = A \cup B \implies A = \emptyset$ or $B = \emptyset$
- X is non-degenerate (only in this talk!)

One-dimensional continua

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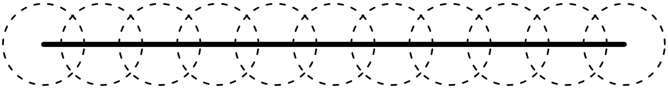
A continuum X is one-dimensional if

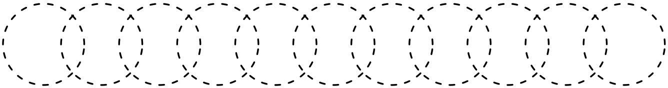
- X has a one-dimensional cover by arbitrarily small open sets
  - a cover $\mathcal{U} = \{U_1, \ldots, U_m\}$ is one-dimensional (1D) if every $x \in X$ belongs to at most 2 sets of $\mathcal{U}$

Arc

Arc

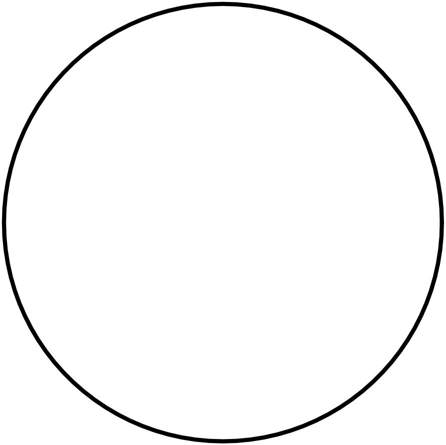


Arc



- chain

Circle



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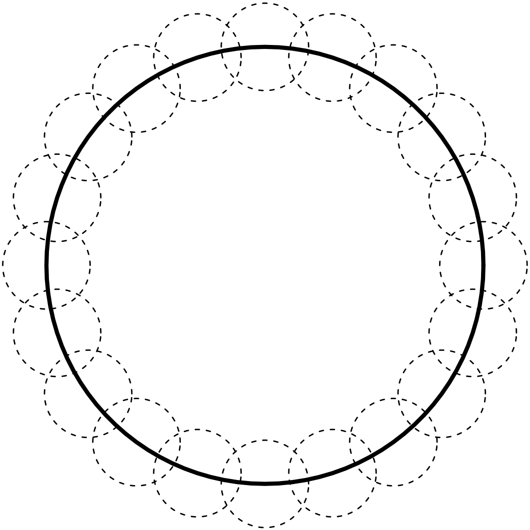

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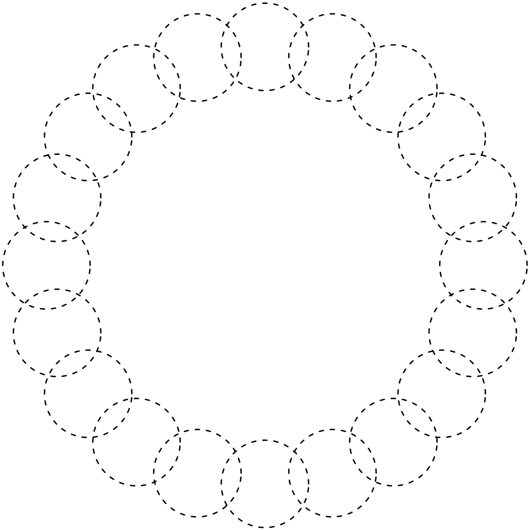
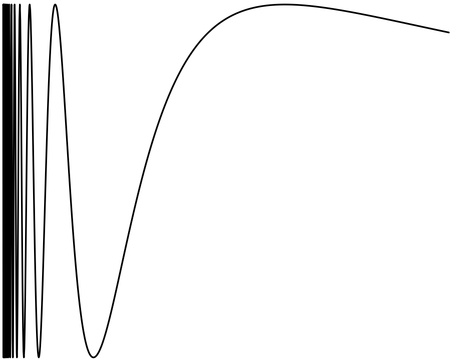


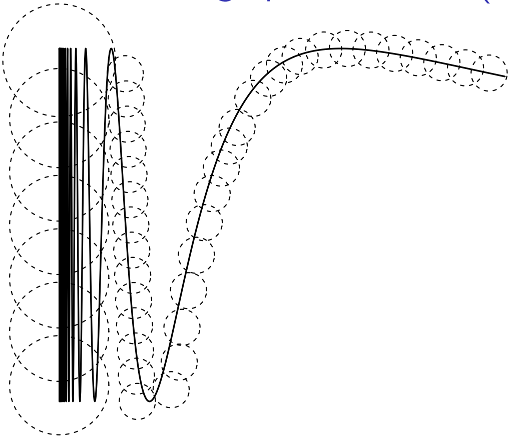
- circle-chain

Topologist's sine curve — graph of $x \mapsto \sin(1/x)$



- not locally connected
- not arcwise connected

Topologist's sine curve — graph of $x \mapsto \sin(1/x)$



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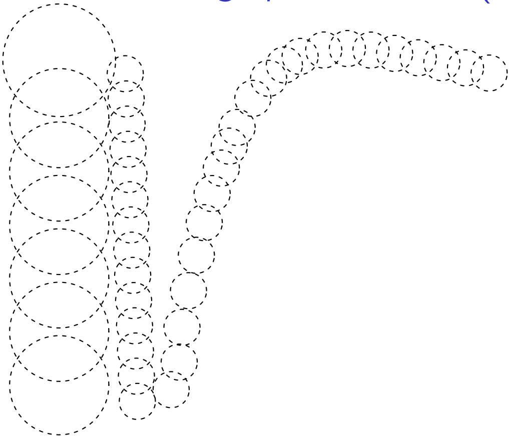
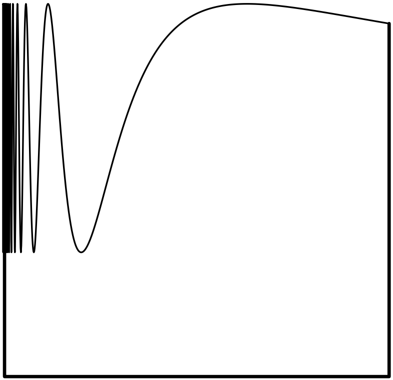


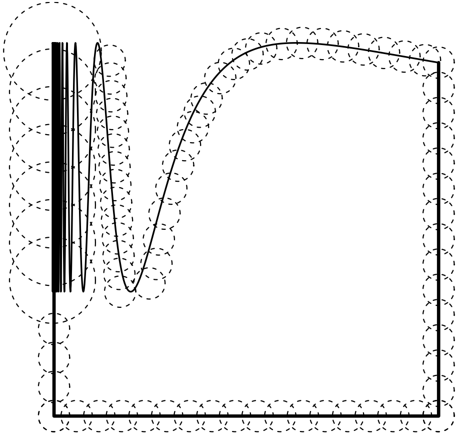
- chain

Warsaw circle



- not locally connected
- arcwise connected
- separates the plane

Warsaw circle



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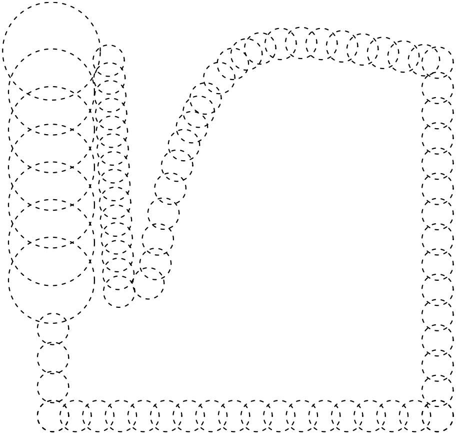

- circle-chain

Arc-like, circle-like and tree-like continua

Nerve $N(\mathcal{U})$ of a 1D cover $\mathcal{U}$: an undirected graph with

- vertices: the sets $U \in \mathcal{U}$
- edges: $\{U, V\}$ with $U \cap V \neq \emptyset$

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A one-dimensional continuum X is

arc-like	$\forall \varepsilon > 0 \; \exists$ 1D ε-cover $\mathcal{U}$ s.t. $N(\mathcal{U})$ is an arc
circle-like	$\forall \varepsilon > 0 \; \exists$ 1D ε-cover $\mathcal{U}$ s.t. $N(\mathcal{U})$ is a circle
tree-like	$\forall \varepsilon > 0 \; \exists$ 1D ε-cover $\mathcal{U}$ s.t. $N(\mathcal{U})$ is a tree

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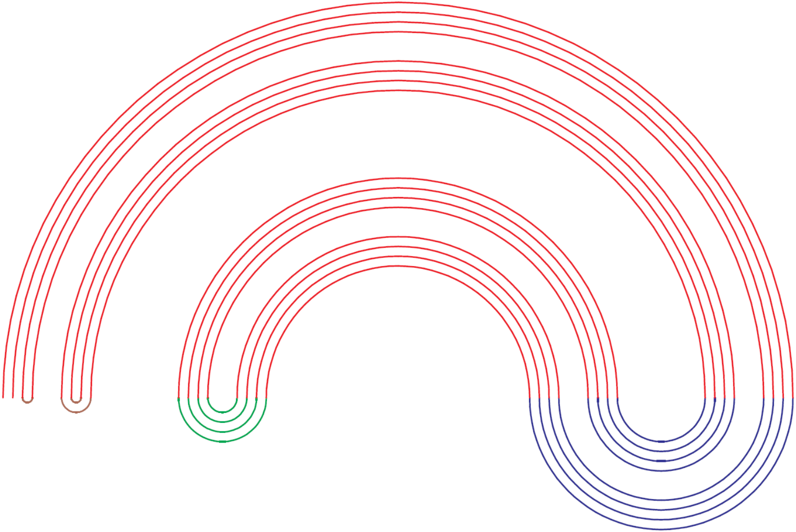
Examples

- arc-like continua: arc, topologist's sine curve
- circle-like continua: circle, Warsaw circle
- tree-like continua: trees, Cantor fan

Knaster buckethandle



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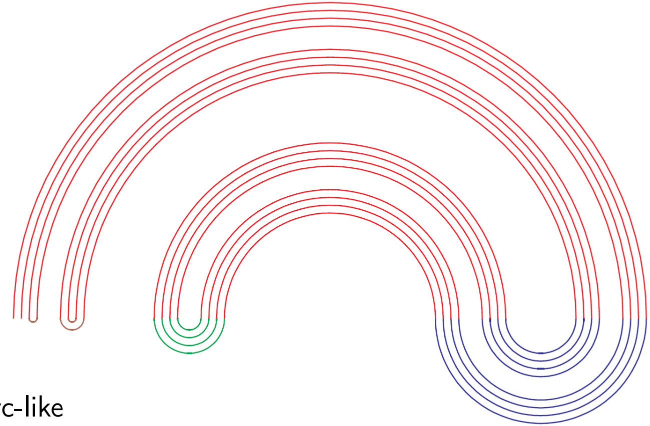
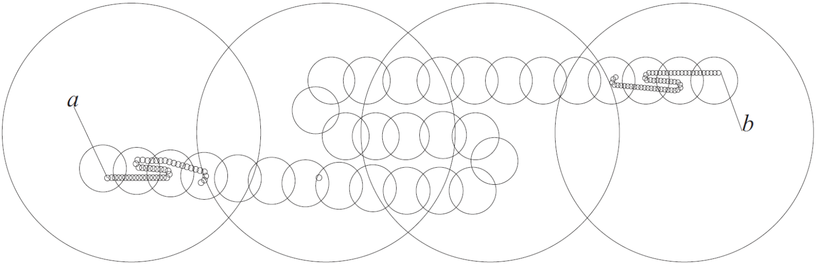
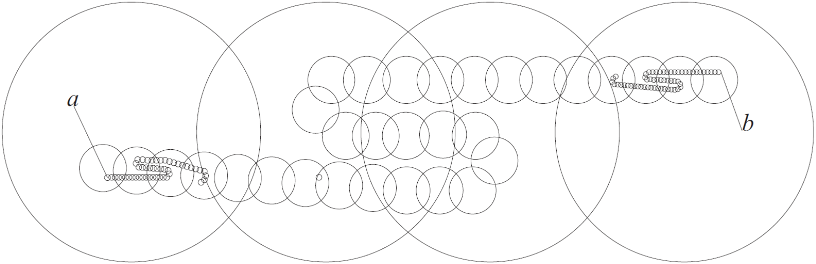


- arc-like
- proper subcontinua: arcs
- indecomposable
  - there are no proper subcontinua A, B s.t. $X = A \cup B$

Pseudo-arc

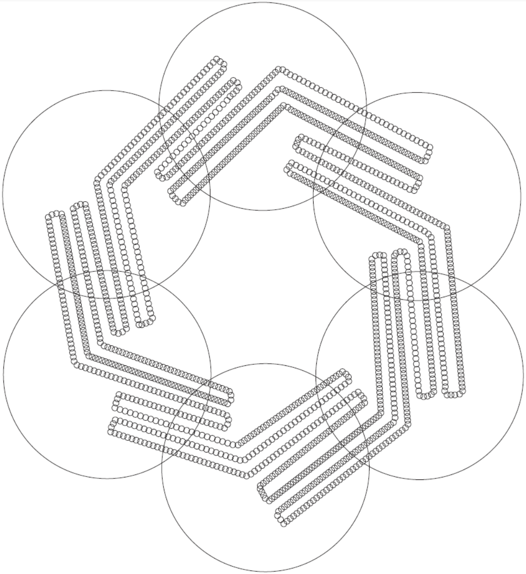


Pseudo-arc

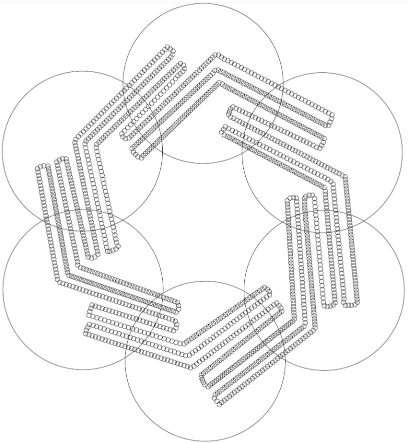


- arc-like
- homogeneous
- proper subcontinua: pseudo-arcs
- hereditarily indecomposable
  - every subcontinuum of the pseudo-arc is indecomposable
- typical continuum in $\mathbb{R}^n$
  - in the space of all subcontinua

Pseudo-circle



Pseudo-circle



- circle-like
- proper subcontinua: pseudo-arcs
- not homogeneous

2. Inverse limits

Cartesian product

- (X_n, d_n) $(n = 1, 2, \dots)$: metric spaces with $\mathrm{diam}(X_n) \leq 1$

Cartesian product $X = \prod_{n=1}^{\infty} X_n$

$$X = \{(x_n)_{n=1}^{\infty} : x_n \in X_n\}$$

Metric d on X:

$$d((x_n)_n, (y_n)_n) = \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{2^n}$$

- every X_n is a compactum $\Rightarrow$ X is a compactum
- every X_n is a continuum $\Rightarrow$ X is a continuum

Inverse limit

Inverse sequence $\{X_n, f_n\}_{n=1}^{\infty}$

- $f_n : X_{n+1} \to X_n$ continuous

$$X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \xleftarrow{f_3} \quad \dots \quad \xleftarrow{f_{n-1}} X_n \xleftarrow{f_n} X_{n+1} \xleftarrow{f_{n+1}} \dots$$

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Inverse limit $X_\infty = \varprojlim (X_n, f_n)$

$$X_\infty = \{(x_n) \in \prod X_n : f_n(x_{n+1}) = x_n \text{ for every } n\}$$

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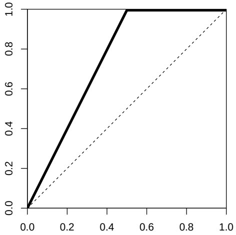
- every X_n is a compactum $\Rightarrow X_\infty$ is a compactum
- every X_n is a continuum $\Rightarrow X_\infty$ is a continuum

Inverse limit — Example 1

$X_n \equiv [0,1] \qquad f_n \equiv f : [0,1] \to [0,1]$

$$f(x) = \begin{cases} 2x & \text{if } x \leq 1/2 \\ 1 & \text{if } x \geq 1/2 \end{cases}$$



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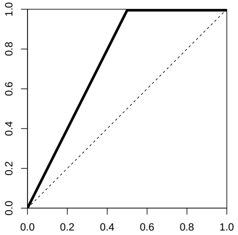


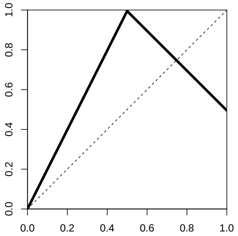
$\varprojlim([0,1], f)$: arc

- the same is true for every monotone f

Inverse limit — Example 2

$$X_n \equiv [0,1] \qquad f_n \equiv f : [0,1] \to [0,1]$$

$$f(x) = \begin{cases} 2x & \text{if } x \leq 1/2 \\ 3/2 - x & \text{if } x \geq 1/2 \end{cases}$$



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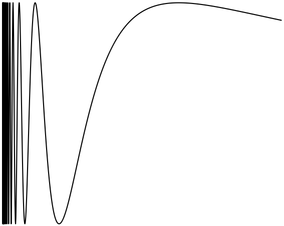
$\varprojlim([0,1], f)$: topologist's sine curve



Inverse limit — Example 3

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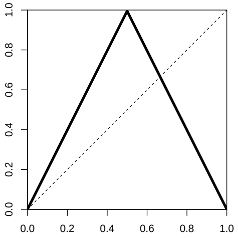

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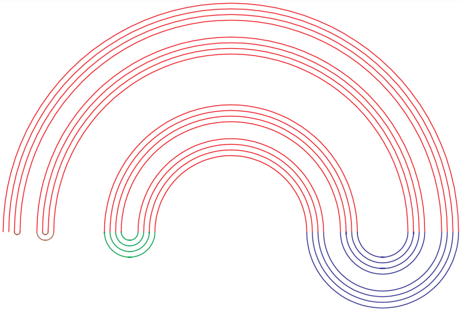
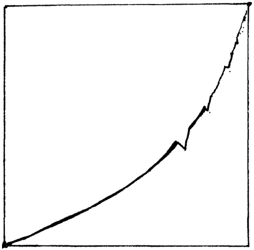
$\varprojlim([0,1], f)$: Knaster buckethandle



Inverse limit — Example 4 (Henderson 1964)

$X_n \equiv [0,1] \qquad f_n \equiv f : [0,1] \to [0,1]$

- C^∞ function constructed as follows:
  - start with $g(x) = x^2$
  - notch its graph with an infinite set of non-intersecting v's which accumulate at $(1,1)$



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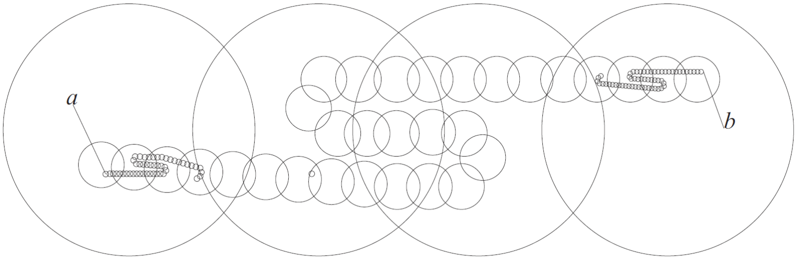
$\varprojlim([0,1], f)$: pseudo-arc



One-dimensional continua $\equiv$ inverse limits of graphs

X is a one-dimensional continuum
$\iff$ there is an inverse sequence $\{X_n, f_n\}$ s.t.

- X_n is a graph
- $f_n : X_{n+1} \to X_n$ is a continuous surjection
- X is homeomorphic to $\varprojlim(X_n, f_n)$

One-dimensional continua ≡ inverse limits of graphs

X is a one-dimensional continuum
$\Longleftrightarrow$ there is an inverse sequence $\{X_n, f_n\}$ s.t.

- X_n is a graph
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- X is homeomorphic to $\varprojlim(X_n, f_n)$

arc-like continua	≡	inverse limits of arcs ($X_n = [0,1]$)
circle-like continua	≡	inverse limits of circles ($X_n = \mathbb{S}^1$)
tree-like continua	≡	inverse limits of trees
one-dim. continua	≡	inverse limits of graphs

3. Applications

Applications

Ingram, Mahavier (2011)

- *Inverse limits have played a crucial role in the development of the theory of continua in the past 50 years or so. Particularly useful is their inherent ability to produce complicated spaces from simple ones...*

Constructions of complicated continua

Anderson, Choquet (1959); Andrews (1961)

- there is a planar tree-like (arc-like) continuum s.t. every subcontinua $A \neq B$ of X are not homeomorphic

Cook (1966)

- there is a continuum X s.t. every continuous $f : X \to X$ is either constant or identity

Bellamy (1979)

- there is a tree-like continuum X without the fixed point property

Simple description of continua

- Knaster buckethandle

$$X = \varprojlim([0,1], f) \qquad f(x) = \begin{cases} 2x & \text{if } x \leq 1/2 \\ 2 - 2x & \text{if } x \geq 1/2 \end{cases}$$

- pseudoarc

$$X = \varprojlim([0,1], f) \qquad f(x) = x^2 - \sum_n g_n(x)$$

- any arc-like continuum

$$X = \varprojlim([0,1], f_n) \qquad f_n : [0,1] \to [0,1]$$

Simplified proofs of properties of continua

Theorem

Knaster buckethandle is indecomposable.

Simplified proofs of properties of continua

Theorem

Knaster buckethandle is indecomposable.

Proof.

- suppose not: $X = A \cup B$, where A, B — proper subcontinua
- A_n, B_n: projections of A, B onto the n-th coordinate
  - A_n, B_n: closed intervals, $A_n \cup B_n = [0, 1]$
- $\exists m$: $A_m \neq [0, 1] \neq B_m$
  - otherwise $A = X$ or $B = X$
- we may assume that $0 \in A_{m+1}$
- then $1/2 \notin A_{m+1}$ and $1 \notin A_{m+1}$
  - otherwise $1 \in A_m \Rightarrow A_m = [0, 1]$
- hence $1/2, 1 \in B_{m+1}$ and so $B_m = [0, 1]$ — a contradiction

Connections with dynamics

Handel (1982)

- the pseudocircle is an attracting minimal set of a plane C^∞ diffeomorphism

Barge, Martin (1990)

- any arc-like continuum $\varprojlim([0,1], f)$ is a global attractor of a plane homeomorphism

Natural extension of a continuous map $f : X \to X$

$$\sigma_f : \varprojlim(X, f) \to \varprojlim(X, f)$$

$$\sigma_f(x_1, x_2, x_3, \dots) = (f(x_1), f(x_2), f(x_3), \dots)$$

- always a homeomorphism
- shares many properties/characteristics with f
  - transitivity, minimality, entropy, ...

4. Open problems

Plane fixed point problem

Does a continuous function taking a non-separating plane continuum into itself always have a fixed point?

- non-separating plane continuum:
  - plane continuum the complement of which is connected
  - intersection of a nested sequence of disks

Plane fixed point problem

Does a continuous function taking a non-separating plane continuum into itself always have a fixed point?

- Bing (1969)
  . . . the most interesting outstanding problem in plane topology.

- Hagopian (1997)
  An affirmative answer would provide a beautiful generalization of the 2D version of Brouwer's fixed point theorem

Plane fixed point problem

Short history

- Ayres (1930)
- Borsuk (1932, 1954)
- Hamilton (1938, 1951)
- Kelley (1939)
- Cartwright, Littlewood (1951)
- Bing (1951, 1969)
- Ward (1959)
- Young (1960)
- Bell (1967, 1978)

Plane fixed point problem

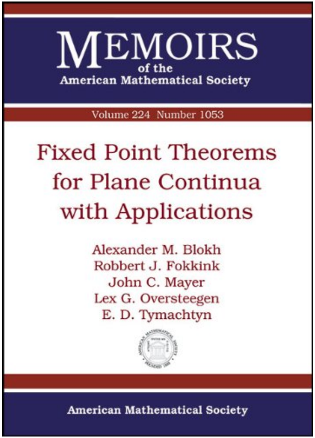
Short history

- Sieklucki (1968)
- Iliadis (1970)
- Hagopian (1971, 1988, 1996, 2007)
- Fugate, Mohler (1977)
- Bellamy (1979)
- Minc (1990, 1999)
- Akis (1999)
- Mayer, Oversteegen, Tymchatyn (2003)

Plane fixed point problem

Short history

- Blokh, Fokkink, Mayer, Oversteegen, Tymchatyn (2013)



Auslander's problem

- Is there a non-separating plane continuum admitting a minimal dynamical system?

Auslander's problem

- Is there a non-separating plane continuum admitting a minimal dynamical system?

Connected question:

- Is there a tree-like continuum admitting a minimal dynamical system?

References

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Thanks for your attention!