How to reconstruct a permutation from a few large patterns

joint work with Erkko Lehtonen

Matej Bel University

February 20, 2024

Introduction

Theory of Permutation Patterns and Patterns Avoidance

- **1935 Erdös - Szekeres** Theorem

  if $a, b \in \mathbb{N}$ then every permutation of rank $(a-1)(b-1)+1$ must contain either the pattern $1\,2\,\ldots\,a$ or the pattern $b\,b-1\,\ldots\,2\,1$

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Starting with the basics: what is a pattern?

- Permutation $\pi \in S_n$ represented as a word

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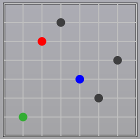
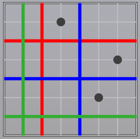
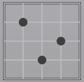
- **Example:** the permutation $3\,1\,2$ is a pattern of $\pi = 1\,5\,6\,3\,2\,4$: it is order isomorphic to ...



The plot of permutation $\pi = 1\,5\,6\,3\,2\,4$





The plot of the pattern $\tau = 3\,1\,2$

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  For $n = 6$

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  and for $n = 4$

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The reconstruction of a permutation from its patterns

- *Is a finite simple graph uniquely determined, up to isomorphism, by the collection of its one-vertex-deleted subgraphs?*

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  How large must n be in order that it is possible to reconstruct π from its $(n-k)$-*deck*? Or from its underlying set?

  In other words, how large must n be in order that the deck is unique to π?

The reconstruction of a permutation from its patterns

- A permutation $\pi \in S_n$ is *reconstructible* from its $(n-k)$-patterns if

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  Mariana Raykova, Permutation reconstruction from minors.

  Rebecca Smith, Permutation reconstruction.

  Both published in *Electron. J. Combin.*, 2006.

  Their work establishes that, **for $\mathbf{n \geq 5}$, every n-permutation is reconstructible from its $\mathbf{(n-1)}$-deck.**

The reconstruction of a permutation from its patterns

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- What if instead we consider $k > 1$? Is a permutation reconstructible from its set of $(n-k)$-patterns or from its $(n-k)$-deck?

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- What is N_k, the least of those numbers N?

  $\mathbf{N_1 = 5}$ and $\mathbf{N_2 = 6}$ (Smith and Raykova respectively)

  $$\mathbf{k + \log_2 k < N_k < \frac{k^2}{4} + 2k + 4} \text{ (Raykova)}$$

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(4)-deck of the permutation

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$$\langle (1423)^4,\ \langle (1432)^4,\ (1234)^1,\ (1243)^5\ \langle (4321)^1 \rangle$$

and it has cardinality $4 + 4 + 1 + 5 + 1 = 15$.

Some open problems

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- What if we do not need all the $(n-k)$-cards?

  In that case what is the value of $\mathbf{H_k(n)}$, the smallest number of cards needed to guarantee the reconstruction?

  Clearly $\mathbf{H_k(n)} \leq \binom{\mathbf{n}}{\mathbf{k}}$.

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- And if $H_k(n)$ is known, how can we recognise a partial deck of a permutation in S_n among the submultisets of cardinality $H_k(n)$ formed by permutations in S_{n-k}?

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Can we find a non-trivial function $f : \{n \in \mathbb{N} \mid n \geq 5\} \to \mathbb{N}$

so that $f(n)$ is the smallest natural number m for which

every permutation $\pi \in S_n$ is uniquely determined by **any** of

its partial $(n-1)$-decks **of cardinality m**?

Searching for ...

- C_n := the largest number for which there exists two distinct permutations with the same $(n-1)$-partial deck of cardinality C_n .

n	C_n
5	$\binom{5}{4} - 1$
6	4
7	5
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- 5 is the unique rank for which the reconstruction of any permutation is not possible from a (proper) partial $(n-1)$-deck.
- $C_n = \lceil n/2 \rceil + 1$ and hence $\mathbf{H(n) = C_n + 1 = \lceil n/2 \rceil + 2}$.

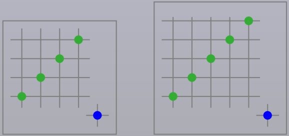
Claiming $\mathbf{H(n)} \geq \lceil \mathbf{n/2} \rceil + \mathbf{2}$.

Take $n \geq 5$,

- for $n = 2m$, the two distinct permutations $(\iota_{m-1} \ominus 1) \oplus \iota_m$ and $(\iota_m \ominus 1) \oplus \iota_{m-1}$ have $\lceil n/2 \rceil + 1$ common $(n-1)$-cards

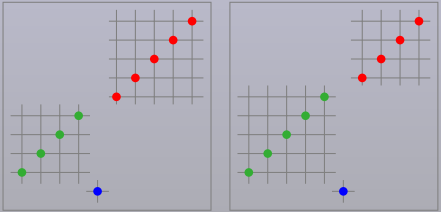
For $\mathbf{m} = \mathbf{5}$:

The skew sums



$\iota_{5-1} \ominus 1$ and $\iota_5 \ominus 1$

The direct sums



$(\iota_4 \ominus 1) \oplus \iota_5$ and $(\iota_5 \ominus 1) \oplus \iota_4$

The decks of these permutations admit the following common submultiset of cardinality $\lceil 10/2 \rceil + 1$:

$$\langle 1\,2\,3\,4\,5\,6\,7\,8\,9, (2\,3\,4\,5\,1\,6\,7\,8\,9)^5 \rangle$$

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- for $n = 2m+1$, the two distinct permutations $\iota_{m-1} \oplus \delta_2 \oplus \iota_m$ and $\iota_m \oplus \delta_2 \oplus \iota_{m-1}$ have $\lceil n/2 \rceil + 1$ common $(n-1)$-cards

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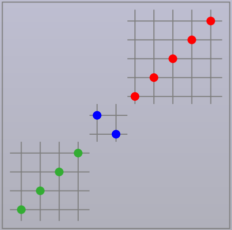

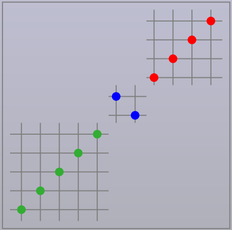

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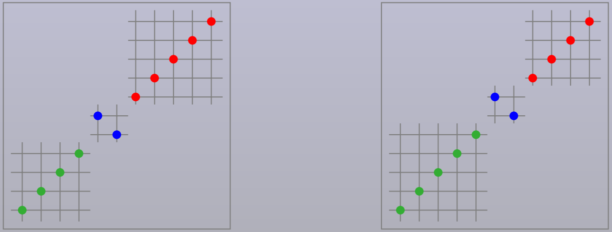

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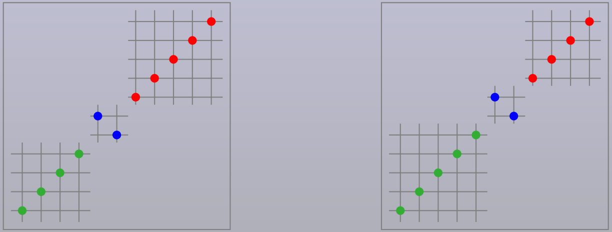

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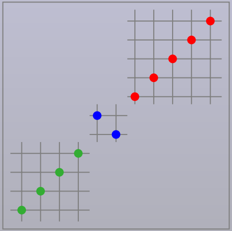

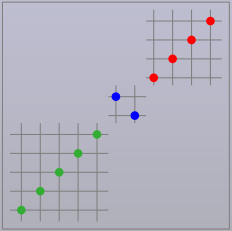

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Conjecture: $H(n) := \lceil n/2 \rceil + 2$ for $n \geqslant 5$.

Proved...

Theorem 19, G. & Lehtonen 2021

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Let $\pi \in S_n$. If $s, t \in [n]$ with $s \leq t$, then $\pi - s = \pi - t$ if and only if $\pi[s, t]$ is a monotone segment in π.

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Clearly $\pi(5) \geq 5$ which implies $\pi = 1\,2\,6'\,5'\,\_\,3\,4$.

Also $\pi(3) \geq 6$ and therefore $\pi = 1\,2\,6'\,5\,\_\,3\,4$. Thus $\pi = 1\,2\,7\,6\,\underline{5}\,3\,4$.

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Ginsburg, Lemma 1(vi) Let $\pi \in S_n$.
If $i, j \in [n]$ with $i < j$, then $(\pi \downarrow j) \downarrow i = (\pi \downarrow i) \downarrow (j - 1)$.

Example:

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$$\begin{array}{ccccc} 1\,2\,6\,5\,3\,4 & \rightsquigarrow & 1\,2\,6\,5\,3 & \rightsquigarrow & 1\,2\,5\,4\,3 = \pi \downarrow 4 \\ 1\,2\,5\,4\,3 & \rightsquigarrow & 1\,5\,4\,3 & \rightsquigarrow & 1\,4\,3\,2 = (\pi \downarrow 4) \downarrow (2) \end{array}$$

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A key idea

Theorem 19, G. & Lehtonen 2021

For $n \geq 5$, every permutation of rank n is reconstructible from $H(n)$ cards.

Theorem's proof

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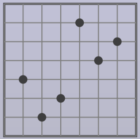
Theorem's proof

- Constructive and it can be turned into a reconstruction algorithm.
- The key idea is to determine $\pi^{-1}(i)$ and $\pi \downarrow i$, for some $i \in [n]$, from the given partial deck of π.
- From the position $\pi^{-1}(i)$ and the pattern $\pi \downarrow i$ it is easy to recover π,:

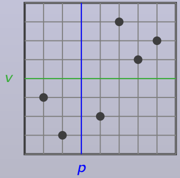
$$\pi = (\pi \downarrow i) \uparrow_{\pi^{-1}(i)} i.$$

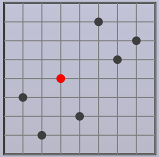
Example: Let $\tau = 3\,1\,2\,6\,4\,5$, $p = 3$ and $v = 4$.

The permutation $\tau \uparrow_p v$ is the permutation we obtain from τ by inserting the value v on position p as illustrated:



The permutation τ





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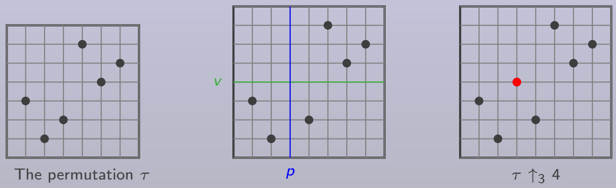

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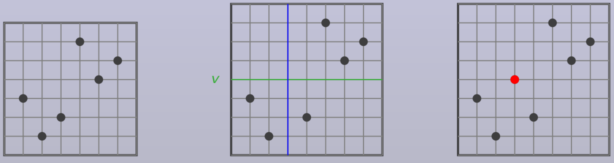

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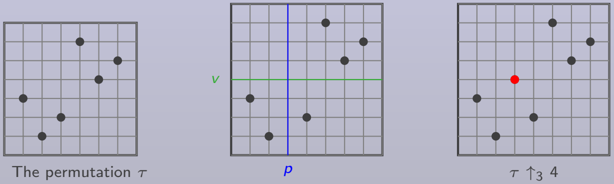

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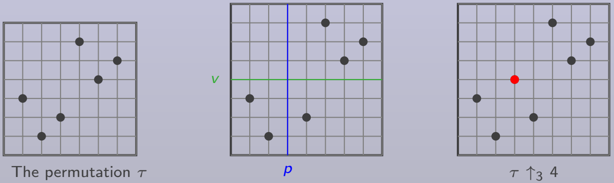

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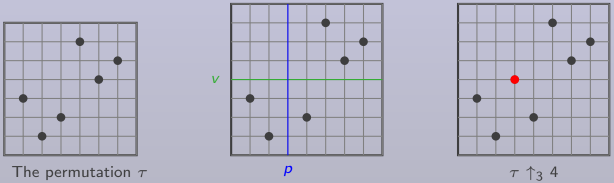

Note: In some cases it is not directly possible to determine $\pi^{-1}(i)$ and $\pi \downarrow i$. Then

- we determine $\pi^{-1}(1)$, which is always possible;
- we build a (partial) deck for $\pi \downarrow 1$ from the partial deck of π;
- we apply the previous procedure aiming now to reconstruct $\pi \downarrow 1$;
- by a recursive application of the algorithm, we end up reconstructing π.

The reconstruction process

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Thus

$$\pi = \tau \uparrow_u \tau(u) \text{ if } \pi[u, v] \text{ is ascending}$$

and

$$\pi = \tau \uparrow_{v+1} \tau(v) \text{ if } \pi[u, v] \text{ is descending.}$$

The reconstruction process: when D contains at least two different cards

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By inspection on D one of two cases must occur:

- **Monotone case:** there is a card of D that contains a monotone sequence $k_1\,k_2\ldots k_s$ such that in every card either $k_1\,k_2\ldots k_s$ or $(k_1-1)\,(k_2-1)\ldots(k_s-1)$ occurs.
- **Non monotone case**

The reconstruction process: the monotone case

- Subcase: D has a unique maximal monotone segment

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Example

$D = \langle (4\,7\,5\,8\,3\,2\,1\,6\,9)^3, 7\,5\,8\,3\,2\,1\,4\,6\,9, 5\,7\,8\,3\,2\,1\,4\,6\,9,$
$5\,7\,6\,8\,3\,2\,1\,4\,9, 5\,8\,6\,9\,3\,2\,1\,4\,7 \rangle.$

The reconstruction process: the monotone case

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The card $\tau = 4\,7\,5\,8\,3\,2\,1\,6\,9$ has a unique maximal monotone segment of length $m = 3$ and multiplicity m.

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Then $\pi = \tau \uparrow_5 4 = \underline{5}\,\underline{8}\,\underline{6}\,\underline{9}\,\underline{4}\,\underline{3}\,\underline{2}\,\underline{1}\,\underline{7}\,\underline{10}$

Example

$D :=$

$\langle (9\,7\,8\,6\,3\,4\,5\,2\,1)^3, (9\,8\,7\,4\,5\,6\,3\,2\,1)^2, 9\,7\,8\,4\,5\,6\,3\,2\,1, 8\,9\,7\,4\,5\,6\,3\,2\,1 \rangle$

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Take

$$a^* := \max A = 4, \qquad b^* := \max B = 5.$$

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$$u := \begin{cases} b^* + 1, & \text{if } |B| = 1 \text{ and } H(n) \leq b^*, \\ b^*, & \text{otherwise,} \end{cases} \qquad = 5$$

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Hence $\pi[5,7] = 4\,5\,6$ is a maximal ascending segment in π.

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$\sigma = \tau[4,6] = 3\,2\,1$ is a maximal monotone segment.

(As we did in a previous case) Apply Ginsburg's Lemma and Proposition 8 to $\pi \downarrow \pi[5,7] \in S_7$ and D' and obtain

$$\pi \downarrow \pi[5,7] = \tau \uparrow_4 (\tau(4)+1) = (6\,4\,5\,3\,2\,1) \uparrow_4 4 = 7\,5\,6\,4\,3\,2\,1.$$

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and now we can immediately reconstruct π:

$$\pi = (\pi \downarrow \pi[5,7]) \uparrow_5 \pi[5,7]$$

$$= (7\,5\,6\,4\,3\,2\,1) \uparrow_5 (4\,5\,6)$$

$$= 10\,8\,9\,7\,4\,5\,6\,3\,2\,1.$$

The reconstruction process: the non monotone case

Step 1 : Determine the position p of 1 in π (and simultaneously the position r of 2 in π).
This is done by comparing the positions of 1 and 2 in the cards and uses Lemmas 13 and 16 (G. & Lehtonen, 2021).

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One of two cases must occur:

- If such a card τ exists then it must be $\pi \downarrow 1$ and then $\pi = \tau \uparrow_p 1$.
- If no such card exists then $\pi \downarrow 1$ is not in the partial deck D.
  Now the strategy is to define a partial deck of $\theta := \pi \downarrow 1 \in S_{n-1}$ by removing 1 from the cards in D.
  We repeat the procedure for θ and D', starting from Step 1.

For more details:

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G. & Lehtonen The Electronic Journal of Combinatorics, 28(3), 2021

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Thank you!