Hexagons and perfect matchings in snarks

An excursion to structural study of snarks

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Multipoles, graphs, matchings

- A multipole M is defined by setting a (finite) vertex set V, a (finite) edge set E, and an incidence relation $I \subseteq V \times E$,
- Every edge $e \in E$ has two ends, each end may (*or may not be*) incident with a vertex $v \in V$, moreover I is injective relation;
- A free end of the edge in M is not incident with a vertex; M may contain *semiedges* (or even *free edges*);
- k-regular multipole – every vertex is incident with **exactly** k edges;
- We will restrict ourselves to 3-regular multipoles;
- A multipole $M(s_1, \ldots, s_n)$ with n free ends is a n-pole;
- A 0-pole is a graph (a cubic graph);
- A subgraph $S \leq G$ is determined by choice of a subset of E (restricting $I\mid_S$);
- A matching M of G is a 1-regular subgraph of G.

The edge-cyclic connectivity $\xi(G)$ of a graph G is the size of the minimum cycle-separating edge-cut.

Let G be a cubic graph:

- There only three cubic graphs without a cycle-separating cut, namely D_3, K_4 and $K_{3,3}$, for them we set $\xi(G) = \beta(G)$,
- The minimal c-s-cut is made of independent edges,
- $\xi(G) \leq \mathrm{girth}(G)$,
- The edge- and vertex-cyclic connectivity coincide,
- The cyclic connectivity of cubic graphs is unbounded (Nedela,Škoviera 1993).

- Let $G = (V, E)$ be a graph (multipole). Having a finite set C we define an edge-colouring $\varphi\colon E \to C$;
- An edge-colouring φ is regular iff it is locally injective at vertices of G;
- A regular edge-colouring of can be regarded as a flow in an abelian group: Kirchhoff laws are satisfied;

For cubic graphs we have

Proposition (Partity Lemma)

Let $M(s_1, \ldots, s_n)$ be a n-pole endowed with a 3-edge-colouring φ. Then the number of free ends of M carrying the same color has the same parity as n.

- The size of minimal C, admitting a regular edge-colouring of G, is called the chromatic index, $\chi(G)$, of G;
- $\chi(G) = \max_{v \in V} \deg(v)$ or $\chi(G) = 1 + \max_{v \in V} \deg(v)$ (Vizing, 1964);
- A snark is a bridgeless (2-connected) cubic graph G with $\xi(G) = 4$.

Theorem (Petersen's Theorem)

*Every **bridgeless** cubic graph admits a perfect matching.*

3-array: G a bridgeless cubic graph, $\mathcal{M} = \{M_1, M_2, M_3\}$ a set of three perfect matchings on G covering its edges, then $E(G)$ consisting of

- E_1: simply covered edges,
- E_0: **uncovered edges**,
- E_2: **doubly covered edges**, and maybe
- E_3: **triply covered edges**.

Definition

A 3-array $\mathcal{M} = \{M_1, M_2, M_3\}$ over G is *optimal* if the subgraph $H = G - E_1$ is minimal. Then $|E_0|$ is the colouring defect of G (the defect of G).

An optimal array is not uniquely determined.

If $\mathrm{df}(G) > 0$, G is a snark. A nontrivial snark is cyclically 4-connected and of girth ≥ 5.

Let $\mathcal{M} = \{M_1, M_2, M_3\}$ be arbitrary 3-array over a cubic graph G.

$\mathrm{core}(\mathcal{M})$ is a subgraph of $G \supseteq H = G - E_1$ consisting of edges of G which are not simply covered:

Proposition (Steffen)

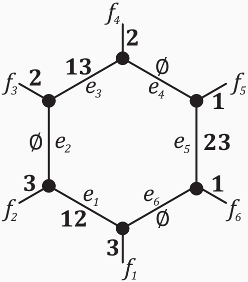
Let $\mathcal{M} = \{M_1, M_2, M_3\}$ be arbitrary 3-array over a cubic graph G. Then

- *nonempty $\mathrm{core}(\mathcal{M})$ has minimum degree 2,*
- *multiply covered edges in $\mathcal{M}$ form a 1-factor of $\mathrm{core}(\mathcal{M})$,*
- *$G - E_0$ is 3-edge-colourable.*

Note that core of an array $\mathcal{M}$ is non-removable subgraph of the snark G.

Theorem (Steffen)

Every snark G has $\mathrm{df}(G) \geq 3$. Moreover, if $\mathrm{df}(G) = 3$, then G contains a non-removable hexagon.



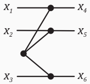
Theorem

Let G be a snark. Then $\mathrm{df}(G) = 3$ if and only if G contains a non-removable hexagon C with complement C' such that its edges connecting C receive colours (112233) or (122331) with respect to the cyclic ordering induced by the order of vertices of C.

- We examine all hexagons C in G and try to extend colourings (112233) or (122331) of dangling edges of $C' = G - C$, ordered in the order "inherited from the ordering of vertices of C".
- We use *brute-force* algorithm to find an optimal array when we have to set defect and defect $\mathrm{df}(G) \geq 4$.

Most of small snarks have defect three;

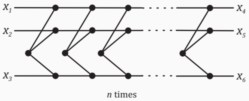
Isaacs (flower) snark J_{2k+1} is a junction of $2k+1$ multipoles Q, for $k>1$

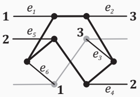


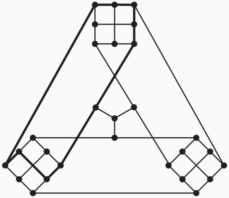
Theorem

$\mathrm{df}(J_{2k+1}) = 3$.

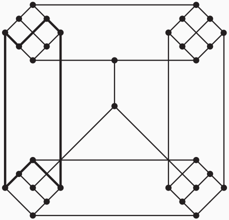
Find the 'right six-cycle'. J_{2k+1} is a junction of Y_n, $n=2k$, and Q



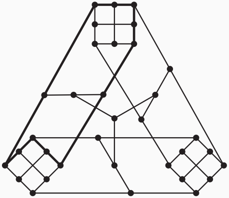




5FLOW28
$\mathrm{df} = 5$



SQUARE32
$\mathrm{df} = 4$



WINDMILL34
$\mathrm{df} = 6$

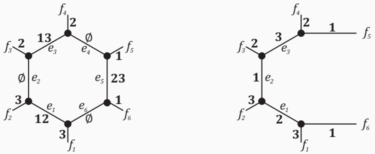
- **5FLOW28** (Máčajova & Raspaud 2005). The smallest (except P_{10}) counterexample to 5-circular-flow hypothesis of Mohar.
- **WINDMILL34** (Brinkman et al. 2013). Strong snark. Smallest snark with perfect matching index 5 (except P_{10}).

A snark G is called strong, if every edge $e \in G$ is supressible.

Theorem (KMNS-red)

Strong snarks have $\mathrm{df} > 5$.

Proof $(\mathrm{df} > 3)$ When $\mathrm{df}(G) = 3$, suppress a doubly covered edge from the optimal core



The proof that $\mathrm{df} > 4$ is a bit longer, based on case-to-case analysis.

Proposition (KMNS-red)

Let G be a snark and with a nonremovable pair of adjacent vertices $\{u, v\}$, and let $\tilde{G}$ be the snark arising from G by inflating both u and v to a triangle. Then $3 \leq \mathrm{df}(\tilde{G}) \leq 4$.

- The resistance $\rho(G)$: the smallest number of edges whose removal from G yields a 3-edge-colourable graph.
- The oddness $\omega(G)$: the minimum number of odd circuits in a 2-factor of G.
- For every snark $\rho(G) \leq \omega(G)$.

Theorem

For every cubic graph G, $\mathrm{df}(G) \geq \rho(G)$.

- There are snarks with arbitrarily large resistance;
- These snarks have oddness 2.

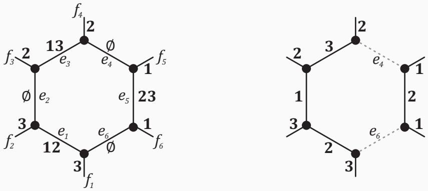
Corollary (KMNS-lgi)

There exist nontrivial snarks with arbitrarily large colouring defect.

Proposition

The resistance of a snark, $\rho(G) = 2$, if and only if the oddness $\omega(G) = 2$.

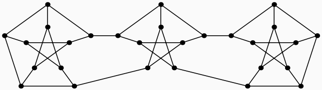
Assume a snark with hexagonal core, then



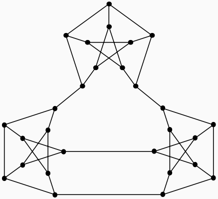
Theorem

If G is a snark with $\mathrm{df}(G) = 3$, then both $\omega(G) = 2$ and $\rho(G) = 2$.

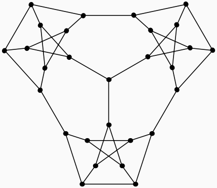
Small snarks with oddness > 2



$\mathrm{df} = 6$



$\mathrm{df} = 7$



$\mathrm{df} = 6$

The smallest snarks with oddness and resistance > 2 (Lukoťka et al., 2015).

All of them are of order 28.

There exists snarks of oddness 2 and arbitrarily large defect

Theorem (Jin-Steffen)

For every snark G, $\mathrm{df}(G) \geq \lceil \mathrm{girth}(G)/2 \rceil$.

Since each vertex of the core H has degree at least 2, H contains a cycle K. Let q be the length of K. At most $\lfloor q/2 \rfloor$ edges of K belong M, so at least $\lceil q/2 \rceil$ edges of K are left uncovered:

$$\mathrm{df}(G) \geq \lceil q/2 \rceil \geq \lceil \mathrm{girth}(G)/2 \rceil,$$

as claimed.

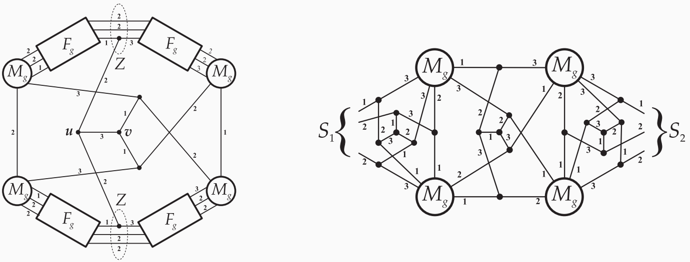
Theorem (KMNS-lgi)

*There exists **nontrivial** snarks of oddness 2 and arbitrarily large girth.*

Corollary

*There exist **nontrivial** snarks with arbitrarily large colouring defect.*

- The snark L with arbitrarily large girth is constructed from the Petersen's graph using superposition:



- A multipole M_g comes from a connected bipartite cubic graph L_g of girth $g = 2r \geq 6$ by removing a path of length 2 ;
- graphs M_g are guaranteed to exist (Nedela and Škoviera, 2001).

Lemma

$\varrho(L) = 2$, *i.e.* $\omega(L) = 2$. *Moreover,* L *is cyclically* 5*-connected.*

- If a cubic graph G is 3-edge-colourable, the set of three perfect matchings covers all the edges of G,
- If G is a snark, a 3-array leave $\mathrm{df}(G)$ edges uncovered,
- The defect $\mathrm{df}(G)$ can be arbitrarily large.

Problem

How many PM's one needs to cover all the edges of a snark?

A *Berge cover* of G is a set $\{M_1, \ldots, M_5\}$ of five perfect matchings, not necessarily distinct, that together cover **all** the edges of G.

Conjecture (Berge's conjecture)

Every bridgeless cubic graph admits a Berge cover.

Even better...

Conjecture (Fulkerson's conjecture)

*A collection of **six** perfect matchings doubly covers the edge set of a cubic graph.*

Recently, Mazzuoccolo proved that the conjectures are equivalent.

Every nontrivial snark G with $\mathrm{df}(G) \leq 4$ has a Berge cover (Seffen, 2015).

Theorem (KMNS-pmi)

Every bridgeless cubic graph with colouring defect 3 admits a Berge cover.

Let $\pi(G)$ be the smallest number of perfect matchings covering all the edges of G, the perfect matching index of G:

Theorem (KMNS-pmi)

Let G be a cyclically 4-edge-connected cubic graph with defect 3. Then $\pi(G) = 4$, unless G is the Petersen graph P.

Snarks with $\pi = 5$ and $\xi = 3$ arise from P and a bipartite graph H as a simple 3-junctions...

Conjecture (Fan-Raspaud conjecture)

Every bridgeless cubic graph contains a triple of perfect matchings M_1, M_2, and M_3 such that $M_1 \cap M_2 \cap M_3 = \emptyset$.

Snarks with $\pi = 4$ admit only cyclic cores and satisfy F-R conjecture automatically...

Nontrivial snarks with defect 3

- G a snark, k-edge-cut R in G with $k \geq 2$,
- R decomposes G into a junction $H * K$ of two k-poles.
- if one of H, K (say H) is a snark, we reduce G getting smaller snark G';
- join the free ends of H by free edges (add one vertex 3-multipole, if needed).

Lemma (KMNS-red)

Let G be a snark with $\mathrm{df}(G) = 3$. If G contains a 2-edge-cut, then G can be reduced to a smaller snark G' with $\mathrm{df}(G') = 3$.

Lemma (KMNS-red)

Let G be a snark with $\mathrm{df}(G) = 3$. If G contains a 4-cycle, then G can be reduced to a smaller snark G' with $\mathrm{df}(G') = 3$.

The proof is rather nontrivial...

Nontrivial snarks with defect 3: Some triangles cannot be avoided

Lemma (KMNS-red)

Let G be a snark with $\mathrm{df}(G) = 3$. If G contains a cycle-separating 3-cut, then G admits a reduction to a smaller snark G' with $\mathrm{df}(G') = 3$, unless one of the resulting components is an ***essential*** *triangle.*

A triangle $T \subseteq G$, where $\mathrm{df}(G) = 3$ is essential in G, if its contraction to a vertex leads to a graph G' such that $\mathrm{df}(G') > 3$.

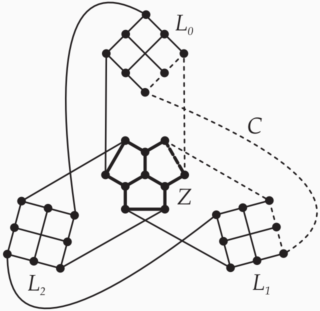
Theorem (Reduction theorem)

For every snark G with $\mathrm{df}(G) = 3$ exactly one of the following holds.

(i) *G has a reduction to a nontrivial snark G' with $\mathrm{df}(G') = 3$.*

(ii) *G arises from a nontrivial snark K with $\mathrm{df}(K) \geq 4$ by inflating a vertex to a triangle; the resulting triangle is essential in G.*

The lesson of all this is that triangles may be important in snarks.



Theorem (KMNS-red)

Let K be a nontrivial snark with $\mathrm{df}(K) \geq 4$, let v be a vertex of K, and let G be the snark created from K by inflating v to a triangle. Then $\mathrm{df}(G) = 3$ if and only if v lies in a heavy cluster of 5-cycles of K, moreover the triangle is essential in G.

Fact

Every irreducible snark of order ≤ 36 has defect three. Known larger irreducible snarks have defect three.

Conjecture

Every irreducible (bicritical) snark G has $\mathrm{df}(G) = 3$.

Even stronger...

Conjecture

Every critical snark G has $\mathrm{df}(G) = 3$.

order	count	values of df	df > 3		loose	2-loose
			girth $\geq$ 4	girth $\geq$ 5		
10	1	3	0	0	1	1
18	2	3	0	0	2	2
20	6	3	0	0	6	1
22	31	3	0	0	31	7
24	155	3	0	0	123	33
26	1297	3	0	0	672	246
28	12517	3,5	0+1+0	0+1+0	4321	1010
30	139854	3,5	0+2+0	0+2+0	33869	10183
32	1764950	3,4,5	1+13+0	1+9+0	273949	88162
34	25286953	3,4,5,6	24+68+7	24+18+7	2390267	888538
36	404899916	3,4,5,6	?	195+304+25	?	?

Snarks from HoG, $\mathrm{cc} \geq 4$, $\mathrm{girth} \geq 4$

$\mathbf{df} > 3$: no 6-cycle C admits an extension of the colouring of the free ends of $G - C$ by either (112233) or (122331);

loose: every 6-cycle admits an extension of the one of (112233) or (122331);

2**-loose:** all 6-cycles admit extensions of both vectors.

Jin-Steffen L. Jin, E. Steffen, *Petersen cores and the oddness of cubic graphs*, J. Graph Theory 84 (2017), 109–120.

KMNS-lgi J. Karabáš, E. Máčajová, R. Nedela, M. Škoviera, *Girth, oddness, and colouring defect of snarks*, Discrete Math. 345 (2022), 113040.

KMNS-pmi J. Karabáš, E. Máčajová, R. Nedela, M. Škoviera, *Berge's conjecture for cubic graphs with small colouring defect*, arXiv:2210.13234[math.CO].

KMNS-red J. Karabáš, E. Máčajová, R. Nedela, M. Škoviera, *Cubic graphs with colouring defect* 3, manuscript.

Steffen E. Steffen, 1-*Factor and cycle covers of cubic graphs*, J. Graph Theory 78 (2015), 195–206.