Skolem sequences and problems of Buratti and Meszka

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Skolem sequences

A sequence $(a_1, a_2, \dots, a_{2n})$ of length $2n$ is a Skolem sequence of order n if it satisfies

1. for every $k \in \{1, 2, \dots, n\}$, there are exactly two elements a_i, a_j of the sequence such that $a_i = a_j = k$, and
2. if $a_i = a_j, i < j$, then $j - i = k$.

Skolem sequences are often given as a collection of pairs $\{(a_i, b_i) : 1 \leq i \leq n\}$ where $\cup_{i=1}^{n}\{a_i, b_i\} = \{1, 2, \dots, 2n\}$.

For example, 42324311 or 4511435232 are Skolem sequences of orders 4 and 5.
These same sequences can be written as $\{(7,8), (2,4), (3,6), (1,5)\}$, and $\{(3,4), (8,10), (6,9), (1,5), (2,7)\}$.

Extended Skolem sequence

A necessary condition for the existence of a Skolem sequence of order n is $n \equiv 0$ or 1 $(mod\ 4)$.

An *extended* Skolem sequence of order n is a sequence $(a_1, a_2, \ldots, a_{2n+1})$ of length $2n + 1$ which satisfies an additional condition that there is exactly one i such that $a_i = 0$.

For example, 5641154623203 or euivalently $\{(4, 5), (9, 11), (10, 13), (3, 7), (1, 6), (2, 8)\}$ is an extended Skolem sequence of order 6.

An extended Skolem sequence with $a_{2n} = 0$ is a *hooked* Skolem sequence.

Existence of Skolem sequences

A Skolem sequence of order n exists whenever $n \equiv 0$ or $1 \pmod 4$.

$n = 4s, n > 4$

$(4s + r - 1, 8s - r + 1) \qquad r = 1, \ldots, 2s$
$(r, 4s - r - 1) \qquad r = 1, \ldots, s - 2$
$(s + r + 1, 3s - r) \qquad r = 1, \ldots, s - 2$
$(s - 1, 3s),\ (s, s + 1),\ (2s, 4s - 1),\ (2s + 1, 6s)$

$n = 4s + 1, n > 5$

$(4s + r + 1, 8s - r + 3) \qquad r = 1, \ldots, 2s$
$(r, 4s - r + 1) \qquad r = 1, \ldots, s$
$(s + r + 2, 3s - r + 1) \qquad r = 1, \ldots, s - 2$
$(s + 1, s + 2),\ (2s + 1, 6s + 2),\ (2s + 2, 4s + 1)$

Hooked Skolem sequences

These exist iff $n \equiv 2$ or 3 $(mod\ 4)$.

$n = 4s + 2,\ n \geq 6$

$(r, 4s - r + 2)$ $\quad r = 1, \ldots, 2s$
$(4s + r + 3, 8s - r + 4)$ $\quad r = 1, \ldots, s - 1$
$(5s + r + 2, 7s - r + 3)$ $\quad r = 1, \ldots, s - 1$
$(2s + 1, 6s + 2),\ (4s + 2, 6s + 3),\ (4s + 3, 8s + 5),\ (7s + 3, 7s + 4)$

$n = 4s - 1,\ n \geq 7$
$(4s + r, 8s - r - 2)$ $\quad r = 1, \ldots, 2s - 2$
$(r, 4s - r - 1)$ $\quad r = 1, \ldots, s - 2$
$(s + r + 1, 3s - r)$ $\quad r = 1, \ldots, s - 2$
$(s - 1, 3s),\ (s, s + 1),\ (2s, 4s - 1),\ (2s + 1, 6s - 1),\ (4s, 8s - 1)$

Number of Skolem sequences

The number N_S of distinct Skolem sequences increases exponentially:

$$N_S \geq 2^{\lfloor \frac{n}{3} \rfloor}$$

n	1	4	5	8	9	12	13	16
N_S	1	6	10	504	2656	455936	3040560	1400156768

Comment: Langford sequences, near-Skolem sequences etc. etc.

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An application of Skolem sequences

Skolem sequences and hooked Skolem sequences can be applied to produce solutions to the

I. Heffter's difference problem :
Partition the elements of the set $\{1, 2, \ldots, 3k\}$, $k \geq 1$, into k ordered triples (a_i, b_i, c_i), $i = 1, 2, \ldots, k$ such that for each i, either $a_i + b_i = c_i$ or $a_i + b_i + c_i = 6k + 1$.

For example, when $k = 2$, then $(1, 3, 4), (2, 5, 6)$ is such a partition of $\{1, 2, 3, 4, 5, 6\}$.

From Skolem sequences to a solution of I.HDP

If $(a_i, b_i), i = 1, \ldots, n$ is a Skolem sequence (or a hooked Skolem sequence) of order n then

$\{(i, a_i + n, b_i + n) : i = 1, \ldots, n\}$ is a solution to the I.HDP.

For example, if $\{(7,8),(2,4),(3,6),(1,5)\}$ is a Skolem sequence of order 4, we obtain from it $\{(1,11,12),\ (2,6,8),\ (3,7,10),\ (4,5,9)\}$ as a solution of I.HDP for $k = 4$.

From I.HDP to cyclic Steiner triple systems

Given a solution to the I.HDP, say, $\{(a_i, b_i, c_i) : i = 1, \ldots, k\}$, we can obtain from it a cyclic Steiner triple system of order $6k + 1$ by taking as base triples

$$\{0, a_i, a_i + b_i\} \textit{ modulo } 6k + 1,$$

meaning that the set of triples is $\{j, j + a_i, j + a_i + b_i\},\ \ j \in Z_{6k+1}$.

For example, our solution $(1, 11, 12), (2, 6, 8), (3, 7, 10), (4, 5, 9)$ of the I.HDP leads to the set of four base triples

$$\{0, 1, 12\},\ \{0, 2, 8\},\ \{0, 3, 10\}, \{0, 4, 9\}$$

each of which represents a set (orbit) of 25 triples, for a total of 100 triples of a Steiner triple system of order 25.

Buratti's problem

K_n is the complete graph on n vertices. The cyclic group Z_n of order n will usually be taken as the set of vertices of K_n. The length of the edge $e = uv,\ u, v \in K_n$ (or the distance between u and v) is given by $d(u, v) = min(|u - v|, n - |u - v|)$. Given a subgraph G of K_n, we denote the (multi)set of edge-lengths of G by $d(G)$:
$d(G) = \{d(e) : e \in G\}$.

BURATTI'S PROBLEM:

Let $p = 2n + 1$ be a prime, let L be any list (=multiset) of $2n$ elements, each from the set $\{1, 2, \ldots, n\}$. Does there exist a Hamiltonian path H in K_p with $V(K_p) = Z_p$ such that the multiset of edge-lengths of H comprises L? (That is, such that $d(H) = L$?)

Some references to Buratti's problem

1. S. Capparelli, A. Del Fra, Hamiltonian paths in the complete graph with edge-lengths 1,2,3, Electronic Journal of Combinatorics 17(2010), R44.

2. P. Horak, A. Rosa, On a problem of Marco Buratti, Electronic Journal of Combinatorics 16(2009), R20.

3. A. Pasotti, M.A. Pellegrini, A new result on the problem of Buratti, Horak and Rosa, Discrete Mathematics (to appear)

Meszka's problem

Quite recently, Mariusz Meszka formulated a very similarly sounding problem:

MP. Let $p = 2n + 1$ be a prime, let L be any list (=multiset) of n elements, $L = \{l_1, l_2, \ldots, l_n\}$, $l_i \in \{1, 2, \ldots, n\}$. Does there exist a near-1-factor F in K_p with $V(K_p) = Z_p$ such that the multiset of edge-lengths of F comprises L?

A near-1-factor consists of n pairwise disjoint edges $\{u_i, v_i\}, i = 1, \ldots, n$ and one isolated vertex.

MESZKA CONJECTURES THAT THE ANSWER IS ALWAYS YES.

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Modified Meszka's problem

Consider the following modification of MP:

MP'. Let $m = 2n + 1$ (an odd number, not necessarily a prime), let $L = \{l_1, l_2, \ldots, l_n\}$, $l_1 \geq l_i$ for $i = 2, 3, \ldots, n$. Does there exist a near-1-factor F in K_m with $V(K_m) = Z_m$ such that the set of edge-lengths of F comprises L?

For primes, MP' is equivalent to MP. For a prime p, MP trivially implies MP', while for a converse, if in a list $L = \{l_1, \ldots, l_n\}$ there is an index $k \neq 1$ such that $l_k > l_1$ and $l_k \geq l_i$ for all $i \neq k$ then the list $qL = \{ql_1, \ldots, ql_n\}$ *mod p* where q is the multiplicative inverse of k (i.e. $k.q = 1$) has $ql_1 \geq ql_i$ for all $i \neq 1$, and L satisfies MP if and only if qL does.

If the list L consists of a_1 1's, a_2 2's,...,a_q q's, we will write $L = \{1^{a_1}, 2^{a_2}, \ldots, q^{a_q}\}$.

A (cyclic) *realization* of a list $L = \{l_1, \ldots, l_n\}$ is a near-1-factor $F = \{\{x_i, y_i\} : i = 1, \ldots, n\} \cup \{0\}$ where $\cup_{i=1}^{n} d(x_i, y_i) = L$.
A realization is *linear* if in the above definition we always have $d(x_i, y_i) = |x_i - y_i|$.

Perfect and almost perfect multisets

A t-set X of nonnegative integers is *perfect* if there exists a sequence S of length $2t$, $S = (s_1, s_2, \ldots, s_{2t})$ such that for each $x \in X$, there is exactly one pair s_i^x, s_j^x such that $s_i^x = s_j^x = x$, and if $s_i^x = s_j^x = x$ with $i < j$ then $j - i = x$; $\{s_i^x, s_j^x\} \cap \{s_k^y, s_l^y\} = \emptyset$, $x, y \in X$, and $\cup_{x \in X}\{s_i^x, s_j^x\} = \{s_1, s_2, \ldots, s_{2t}\}$.

The definition of a perfect multiset is similar.

Almost perfect t-sets or t- multisets are defined similarly; such sequences have length $2t + 1$, with the second-last term equal to 0.

For example, $\{2, 3, 4\}$ and $\{2, 3^2, 6\}$ are perfect since 423243 and $62323'363'$ are sequences with required properties.

Examples of almost perfect multisets are $\{2, 4, 5, 6\}$ and $\{3, 4, 5^2\}$, as shown by the corresponding sequences 456242506 and $5345'35405'$.

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Two observations

An observation:

For each $x \geq 1$, the multiset $\{x^x\}$ is perfect.

This is obvious, as the sequence $x', x'', \ldots, x^{(x)}, x', x'', \ldots, x^{(x)}$ shows.

An important corollary: it suffices to consider lists
$L = \{1^{a_1}, 2^{a_2}, \ldots, n^{a_n}\}$ in which $a_i \leq i - 1$ for all $i = 2, 3, \ldots, n$.

Another observation:

If A, B are two multisets and both are perfect or both are almost perfect then $A \cup B$ is perfect.
If one is perfect and the other is almost perfect then $A \cup B$ is almost perfect.

Results

Theorem 1. Meszka's conjecture holds for any list $L = \{1^{a_1}, 2^{a_2}, 3^{a_3}\}$.

Theorem 2. Meszka's conjecture holds for any list $L = \{1^{a_1}, 2^{a_2}, 3^{a_3}, 4^{a_4}\}$.

Theorem 3. Meszka's conjecture holds for any list $L = \{1^{a_1}, 2^{a_2}, 3^{a_3}, 4^{a_4}, 5^{a_5}$.

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One more result

THEOREM. Let $L = \{1^{a_1}, 2^{a_2}, \ldots, n^{a_n}\}$. For each $a_1 \geq \frac{n^2}{2}$, L is perfect or almost perfect.

THANK YOU FOR YOUR ATTENTION !