O aplikáciách po-grup a ℓ-grúp v niektorých algebrických a kvantových štruktúrach

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The talk given at Semninár Aká si mi krásna, Oct. 23, 2012, FPV UMB, B. Bystrica, Slovakia

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- effect algebras
- MV-algebras - compatibility

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- state or FAS on an algebraic structure $(M; +, ', 0, 1)$, $s: \ M \to [0,1]$ (i) $s(1) = 1$, (ii) $s(a+b) = s(a) + s(b)$ if $a + b \in M$

po-groups and ℓ-group

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- normal ideal $z + H = H + z$

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- Boolean algebra, OML: $a + b\ \exists$ iff $a \leq b'$, $a + b := a \vee b$

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- (RDP): If $c \leq a + b$ $\exists$ $a_1, b_1 \in M$ such that $a_1 \leq a$, $b_1 \leq b$ and $c = a_1 + b_1$.
- $a_1 + a_2 = b_1 + b_2$, $\exists$ $c_{11}, c_{12}, c_{21}, c_{22} \in M$ s.t. $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$, and $b_2 = c_{21} + c_{22}$.

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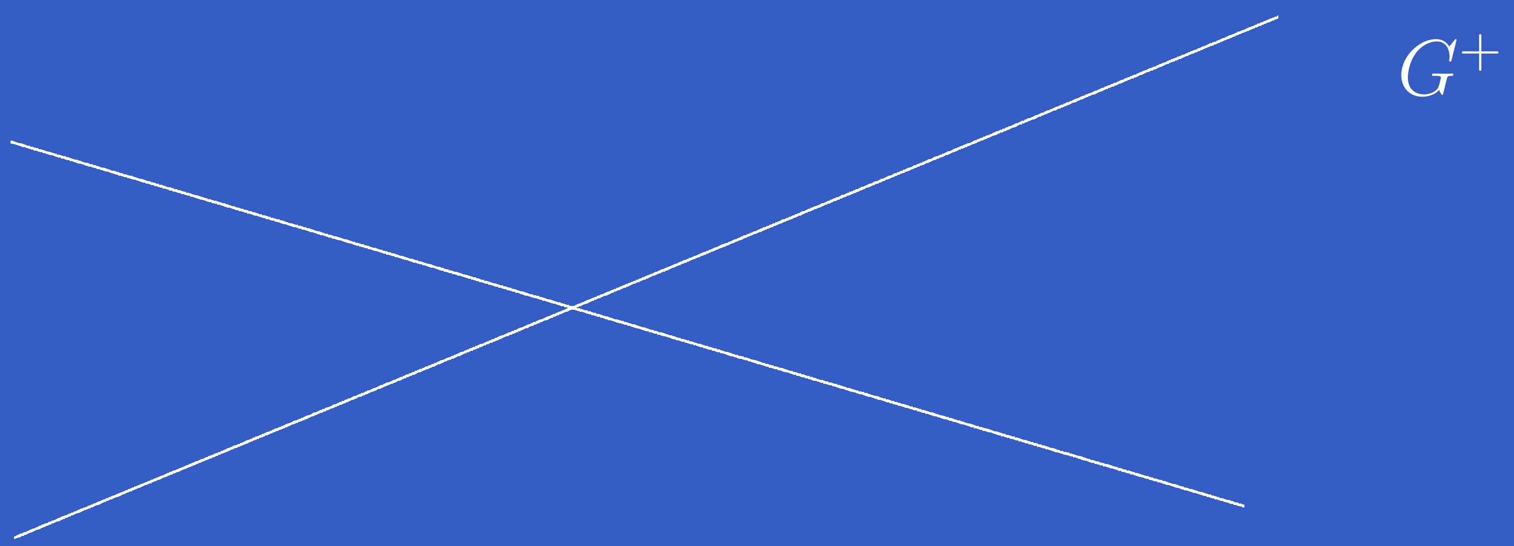
- s state on (G,u): $s(u)=1$, $s(G^+) \subseteq \mathbb{R}^+$, $s(g+h) = s(g) + (h)$. $\mathcal{S}(\Gamma(G,u)) \cong \mathcal{S}(G,u)$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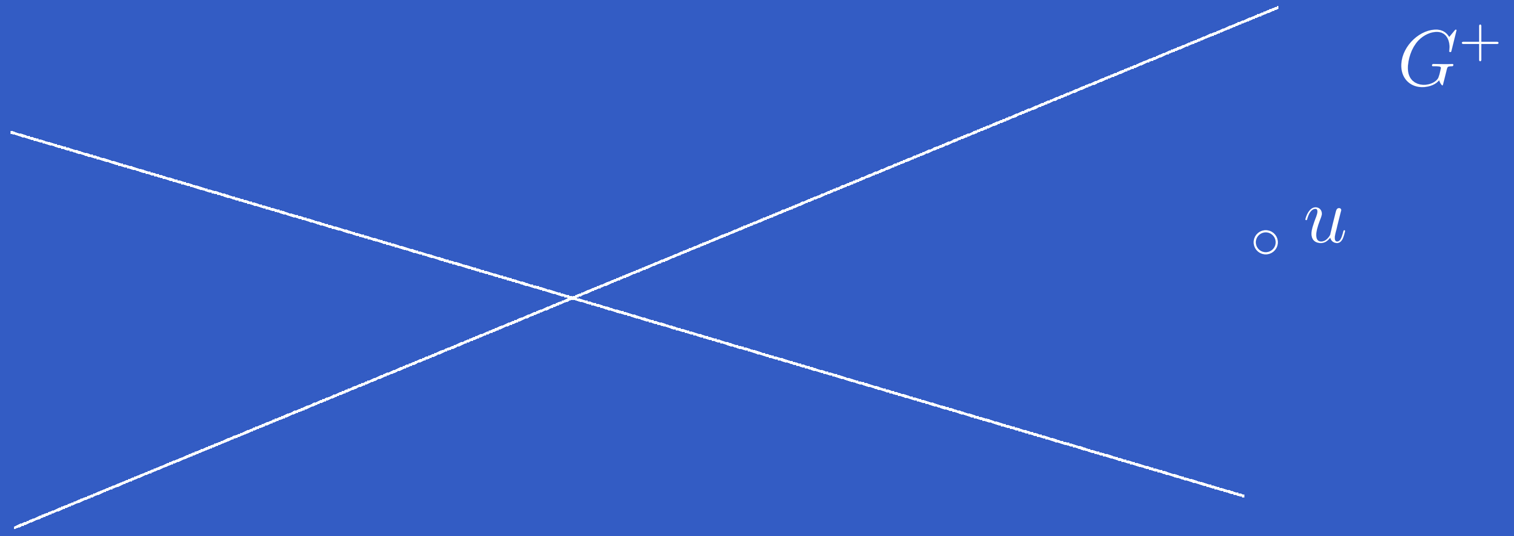
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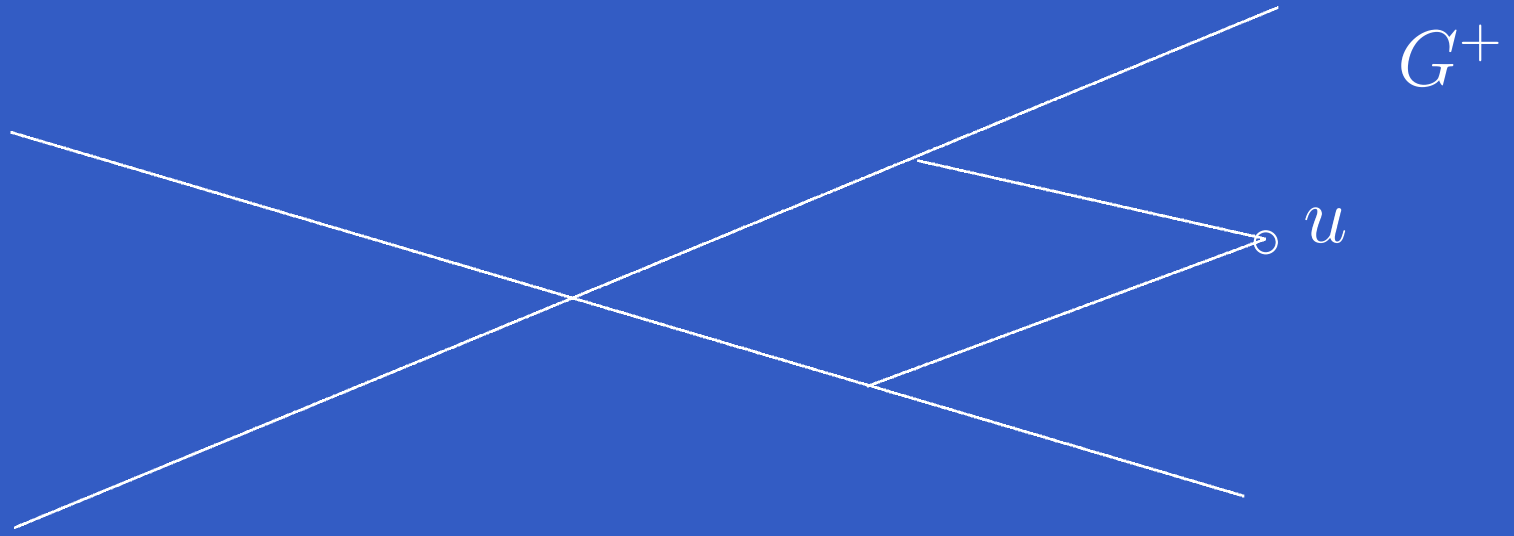
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- There is a categorical equivalence between the category perfect EAs with RDP and the variety of Abelian po-groups

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- Di Nola-Lettieri: the variety of perfect MV-algebras is categorically equivalent with the category of Abelian ℓ-groups
- Komori: The lattice of varieties of MV-algebras is countable

GMV-algebras - pseudo MV-algebras

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- GMV-non commutative generalization of MV-algebras

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z;$

(A2) $x \oplus 0 = 0 \oplus x = x;$

(A3) $x \oplus 1 = 1 \oplus x = 1;$

(A4) $1^{\sim} = 0; 1^{-} = 0;$

(A5) $(x^{-} \oplus y^{-})^{\sim} = (x^{\sim} \oplus y^{\sim})^{-};$

(A6) $x \oplus (x^{\sim} \odot y) = y \oplus (y^{\sim} \odot x) = (x \odot y^{-}) \oplus y = (y \odot x^{-}) \oplus x;$

(A7) $x \odot (x^{-} \oplus y) = (x \oplus y^{\sim}) \odot y;$

(A8) $(x^{-})^{\sim} = x.$

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- $x \vee y = x \oplus (x^\sim \odot y)$ and $x \wedge y = x \odot (x^- \oplus y)$.
- GMV-algebra M is an MV-algebra iff $x \oplus y = y \oplus x$ for all $x, y \in M$.

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- Dvurečenskij: The variety of GMV-algebras is categorically equivalent with the category of unital ℓ-groups not necessarily Abelian

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- AD+ Holland: The lattice of subvarieties of GMV-algebras is uncountable
- The class of GMV-algebras s.t. every maximal ideal is normal is a very large variety

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- 1-1 correspondence $\mathcal{S}(\Gamma(G, u))$ and $\mathcal{S}(G, u)$
- Let u be the translation $u(t) = t + 1$, $t \in \mathbb{R}$,

$$\mathrm{BAut}(\mathbb{R}) = \{g \in \text{Aut}(\mathbb{R}) :\ \exists\, n \in \mathbb{N},\ \ u^{-n} \leq g \leq u^n\}.$$

Then $\Gamma(\text{BAut}(\mathbb{R}), u))$ is stateless and it generates the variety of GMV-algebras

Top varieties

For any value V of (G, u), we set

$$K(V) = \bigcap_{g \in G} g^{-1} V g$$

(we momentarily employ multiplicative notation for (G, u)). Then $K(V)$ is a normal convex ℓ-subgroup of (G, u) contained in V, and $(G/K(V), G/V)$ is a primitive transitive ℓ-permutation group, called a **component** of G.

Let

$$\mathcal{T}(\mathcal{V}) = \{\Gamma(G,u) :\ \Gamma(G/K(V), u/K(V)) \in \mathcal{V},\ \forall\, V \in \Gamma(u)\}.$$

By [DvHo], $\mathcal{T}(\mathcal{V})$ is a variety, referred to as a **top variety** of $\mathcal{V}$.

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$$\mathcal{T}(\mathcal{S}_n) = \mathcal{T}(\mathcal{S}_n^\omega) =$$
$$=: \mathcal{BP}_n = \mathcal{T}(\mathcal{BP}_n).$$

Komori characterization

$$\delta(i) = \{n \in \mathbb{N} : \ 1 \leq n, \ n \text{ is a divisor of } i\},$$

$J \subset \mathbb{N}$ and $i \geq 2$ we let

$$\Delta(i, J) = \{d \in \delta(i) \setminus \bigcup_{j \in J} \delta(j)\}.$$

In case $J = \emptyset$, we define $\Delta(i, \emptyset) = \delta(i)$.

Lettieri Di Nola: $\mathcal{V}$ there exist finite sets I and J, $M \in \mathcal{V}$ iff M satisfies the identities

$$((n+1) \odot x^n)^2 = 2 \odot x^{n+1}, \quad n = \max\{I \cup J\}, \tag{3.2}$$

$$(p \odot x^{p-1})^{n+1} = (n+1) \odot x^p, \tag{3.3}$$

for every integer p, $1 < p < n$, such that p is not a divisor of any $i \in I \cup J$, and

$$(n+1) \odot x^q = (n+2) \odot x^q \tag{3.4}$$

for every $q \in \bigcup_{i \in I} \Delta(i, J)$.

If $I = \emptyset$, we rewrite (3.2)–(3.3) as follows: Let J be a nonempty finite set of positive integers, and consider the identity

$$((n+1) \odot x^n)^2 = 2 \odot x^{n+1}, \quad n = \max J, \qquad (3.6)$$

as well as all identities of the form

$$(p \odot x^{p-1})^{n+1} = (n+1) \odot x^p \qquad (3.7)$$

where p is an integer with $1 < p < n$ and p is not a divisor of any $i \in J$.

Theorem 0.1 *Let $\mathcal{W}$ be the variety of GMV-algebras from $\mathcal{M}$ satisfying identities* (3.6)–(3.7) *for some finite nonempty set J of natural numbers and for all integers p, $1 < p < n$, such that p is not a divisor of any $j \in J$. Then*

$$\mathcal{W} = \bigvee_{j \in J} \mathcal{BP}_j = \bigvee_{j \in J} \mathcal{T}(\mathrm{V}(S_j^{\omega} :\ j \in J)).$$

Theorem 0.2 *Let $\mathcal{W}$ be a proper top subvariety of the variety $\mathcal{M}$. Then there exists a finite set J of natural numbers such that*

$$\mathcal{W} = \bigvee_{n \in J} \mathcal{BP}_n. \tag{3.8}$$

In addition, a GMV-algebra $M \in \mathcal{M}$ belongs to $\mathcal{W}$ if and only if M satisfies identities (3.6)–(3.7) *for any $n \in J$ and for any integer p, $1 < p < n$, such that p is not a divisor of any $n \in J$.*

- **Theorem 0.3** *There are only countably many proper top subvarieties of GMV-algebras within the variety* $\mathcal{M}$.

- **Theorem 0.4** *There are only countably many proper top subvarieties of GMV-algebras within the variety* $\mathcal{M}$.

Covers of the Variety of MV-algebras

Aim: To describe all covers of the variety of MV-algebras, $\mathcal{MV}$ which are in $\mathcal{SYM} \cap \mathcal{M}$.

Covers of the Variety of MV-algebras

Aim: To describe all covers of the variety of MV-algebras, $\mathcal{MV}$ which are in $\mathcal{SYM} \cap \mathcal{M}$.

- Holland 2004 found some non-commutative covers of the variety of Boolean subalgebras, i.e., generated by $\Gamma(\mathbb{Z}, 1)$.

Covers where at least one element has a noncommutative radical

Covers where at least one element has a noncommutative radical

Theorem 0.6 *If $\mathcal{G}$ is a cover of the variety of Abelian ℓ-groups, $\mathcal{A}$, then the variety $\mathcal{MV} \vee \Phi(\mathcal{G}) \subseteq \mathcal{SYM} \cap \mathcal{M}$ is a cover of the variety of MV-algebras, $\mathcal{MV}$, such that at least one its element has a non-commutative radical.*

Covers where at least one element has a noncommutative radical

Theorem 0.7 *If $\mathcal{G}$ is a cover of the variety of Abelian ℓ-groups, $\mathcal{A}$, then the variety $\mathcal{MV} \vee \Phi(\mathcal{G}) \subseteq \mathcal{SYM} \cap \mathcal{M}$ is a cover of the variety of MV-algebras, $\mathcal{MV}$, such that at least one its element has a non-commutative radical.*
And conversely.

- Holland-Medvedev 1994 – there is uncountably many covers of the variety of Abelian ℓ-groups.

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Corollary 0.9 *$\mathcal{MV}$ has uncountably many covers in $\mathcal{SYM} \cap \mathcal{M}$.*

Covers where each element has a commutative radical

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- p - prime, the **Scrimger group** S_p

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Covers where each element has a commutative radical

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  where $\mathbb{Z}_i = \mathbb{Z}$

- $\mathcal{S}_p$ the variety of ℓ-groups generated by S_p. If $p \neq q$, then $\mathcal{S}_p \neq \mathcal{S}_q$. $\mathcal{S}_p$ is a cover of $\mathcal{A}$.

Theorem 0.10 *Let p be any prime number, $n \geq 1$ an integer, S_p be the Scrimger ℓ-group with a fixed strong unit $u_n = (p^n; 0, \ldots, 0)$, and let $\Sigma(S_p, n)$ be the variety of symmetric GMV-algebras in $\mathcal{M}$ generated by the p^n-perfect GMV-algebra $\Gamma(S_p, u_n)$. Then $\mathcal{MV} \vee \Sigma(S_p, n)$ is a cover of $\mathcal{MV}$ such that* **every element of** $\Sigma(S_p, n)$ **has a commutative radical**.

Theorem 0.11 *Let p be any prime number, $n \geq 1$ an integer, S_p be the Scrimger ℓ-group with a fixed strong unit $u_n = (p^n; 0, \ldots, 0)$, and let $\Sigma(S_p, n)$ be the variety of symmetric GMV-algebras in $\mathcal{M}$ generated by the p^n-perfect GMV-algebra $\Gamma(S_p, u_n)$. Then $\mathcal{MV} \vee \Sigma(S_p, n)$ is a cover of $\mathcal{MV}$ such that* **every element of** $\Sigma(S_p, n)$ **has a commutative radical**.

And conversely

Remarks

(1) If $\mathcal{V}$ is a non-commutative cover of the variety of Boolean algebras, $\mathcal{B}$, then $\mathcal{MV} \cap \mathcal{V} = \mathcal{B}$, and $\mathcal{MV} \vee \mathcal{V}$ is a cover of $\mathcal{MV}$. (The converse is not true)

(2) Holland's example

$$T = \{\sum m_i t^{n_i} : \ m_i, n_i \in \mathbb{Z}\}$$

$$(r, n)(s, m) = (r + t^n s, n + m)$$

$$S_t = T \overleftarrow{\times} \mathbb{Z}, \quad \mathcal{C}_t = \Gamma(\Gamma(S_t, (1, 0)))$$

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$\mathcal{C}_t$ and $\mathcal{C}_t \vee \mathcal{MV}$, $t \in \mathbb{R}^+$, are non-commutative covers of $\mathcal{B}$ and $\mathcal{MV}$,
We have continuum many covers, $\mathcal{MV} \vee \mathcal{C}_t$, of $\mathcal{MV}$ which are not symmetric but they are from $\mathcal{M}$.

Pseudo Effect Algebras

- AD+ Vetterlein- noncommutative generalization of EAs

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- If $a+b$ exists, there are elements $d, e \in E$ such that $a+b = d+a = b+e$.
- If $a+b$ and $a+c$ exist and are equal, then $b = c$. If $b+a$ and $c+a$ exist and are equal, then $b = c$.

Pseudo Effect Algebras

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- $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exist, and in this case, $(a + b) + c = a + (b + c)$.
- If $a + b$ exists, there are elements $d, e \in E$ such that $a + b = d + a = b + e$.
- If $a + b$ and $a + c$ exist and are equal, then $b = c$. If $b + a$ and $c + a$ exist and are equal, then $b = c$.
- If $a + b$ exists and $a + b = 0$, then $a = b = 0$.

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- RDP: $a_1 + a_2 = b_1 + b_2$, there are four elements $c_{11}, c_{12}, c_{21}, c_{22}$ such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$, and $b_2 = c_{21} + c_{22}$.

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- (RDP)$_1$: RDP + $x \leqslant c_{12}$ and $y \leqslant c_{21}$, we have $x + y$, $y + x$ exists in E and $x + y = y + x$,

- RDP$_2$: RDP + $d_2 \wedge d_3 = 0$ - pseudo MV-algebra

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- AD+Vetterlein: The category of pseudo effect algebras with RDP$_1$ is categorically equivalent with the category of unital po-group with RDP$_1$

Pseudo BL-algebras

- pseudo BL-algebra - an algebra
  $M = (M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ $\langle 2, 2, 2, 2, 2, 0, 0 \rangle$

Pseudo BL-algebras

- pseudo BL-algebra - an algebra $M = (M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ $\langle 2, 2, 2, 2, 2, 0, 0 \rangle$
- (i) $(M; \odot, 1)$ is a monoid (not neces. comm.), $\odot$ is associative with neutral element 1.

  (ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice.

  (iii) $x \odot y \le z$ iff $x \le y \to z$ iff $y \le x \rightsquigarrow z$ $x, y \in M$.

  (iv) $(x \to y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x),\ x, y \in M.$

  (v) $(x \to y) \vee (y \to x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow y),\ x, y \in M.$

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- *every linear PBL is good, (representable)*

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- $(G^+)^J \uplus (G^-)^I$
- multiplication is defined as follows:

$$\langle a_i^{-1} : i \in I\rangle \cdot \langle b_i^{-1} : i \in I\rangle = \langle (b_i a_i)^{-1} : i \in I\rangle$$

$$\langle a_i^{-1} : i \in I\rangle \cdot \langle f_j : j \in J\rangle = \langle a_{\lambda(j)}^{-1} f_j \vee e : j \in J\rangle$$

$$\langle f_j : j \in J\rangle \cdot \langle a_i^{-1} : i \in I\rangle = \langle f_j a_{\rho(j)}^{-1} \vee e : j \in J\rangle$$

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- similarly for other connectives
- $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ is a pseudo BL-algebra

- $\mathbb{Z} + \uplus(\mathbb{Z} + \mathbb{Z}^- \times \mathbb{Z}^-)$ is a pseudo BL-algebra which is not good

Extremal State MV-algebras

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- s state on M. $\sigma_s(\alpha, a) = \alpha\, s(a) \otimes 1$ on $[0,1] \otimes M$. (M, σ_s) is ex.s.MV iff s is extremal
- $M = [0,1]^2$, $\sigma_0(x, y) = (x, x)$, $(x, y) \in [0,1]^2$.

- $[0,1]^2$ with the diagonal operator σ_0 generates the variety of state morphism MV-algebras

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- $[0,1]^2$ with the diagonal operator σ_0 generates the variety of state morphism MV-algebras
- The lattice of varieties of state morphism MV-algebras is uncountable
- What are generators of MV-algebras with state operator ? -Open problem

Thank you for your attention

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