Oscilace nelineárních rovnic: spojité versus diskrétní modely

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Motivation

Why second order differential equations?

- mathematical models for many real world problems
- science, technology, economics, psychology, defense and demography
- Euler-Lagrange equations of energy functionals

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Why second order differential equations?

- mathematical models for many real world problems
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Many simple differential equations cannot be solved explicitly

↓

existence of solutions
asymptotic properties of solutions

↓

boundedness, convergence to zero, oscillation

Continuous models: PDE's

- Laplace type equation

$$\Delta u + F(|x|, u) = 0$$

$$\Delta u = \operatorname{div} \operatorname{grad} u = \langle \nabla, \nabla u \rangle = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$$

$u = u(x)$ is a radially symmetric solution $\Longleftrightarrow$ $y = y(r) = u(|x|)$ is a solution of equation

$$\big(r\, y'(r)\big)' + b(r) F(y(r)) = 0 \qquad (r \ge c).$$

- PDE with p-Laplacian

$$\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) + F(|x|, u) = 0$$

Discrete models: recurrence relations

Fibonacci numbers: $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

Fibonacci recurrence relation

$$x_{n+2} = x_{n+1} + x_n \qquad (n \in \mathbb{N})$$

with the initial conditions

$$x_1 = 1, \quad x_2 = 1.$$

Denote the forward difference operator

$$\Delta x_n = x_{n+1} - x_n.$$

Fibonacci recurrence relation can be written as

$$\Delta((-1)^n \Delta x_n) + (-1)^n x_{n+1} = 0.$$

Outline of the talk

Equations with the Sturm-Liouville operator

Linear differential equation

$$x'' + b(t)x = 0,$$

$$\big(a(t)x'\big)' + b(t)x = 0, \tag{L}$$

where a, b are positive continuous functions for $t \geq 0$.

Emden-Fowler differential equation

$$\big(a(t)x'\big)' + b(t)|x|^\beta \operatorname{sgn} x = 0, \quad \beta \neq 1, \beta > 0. \tag{EF}$$

$\beta > 1$: super-linear equation, $\beta < 1$: sub-linear equation.

- ↪ Emden (1907): Gaskugeln, Anwendungen der mechanischen Warmentheorie auf Kosmologie und metheorologische Probleme, Leipzig.
- ↪ Fowler (1930): The solutions of Emden's and similar differential equations, Monthly Notices Roy. Astronom. Soc.

A solution is *oscillatory* if it has a sequence of zeros converging to infinity, otherwise it is said to be *nonoscillatory*.

Equation is *oscillatory* if any its solution is oscillatory.

Linear differential equations – Sturmian theory

all sols are oscillatory
all sols are nonoscillatory

Nonlinear differential equations

all sols are oscillatory
all sols are nonoscillatory
there exist oscillatory and nonoscillatory solutions

Oscillation criteria for Emden-Fowler equation (EF):

$\beta > 1$ (case $a(t) \equiv 1$ by Atkinson 1955)

$$\int_0^\infty b(t) \int_0^t \frac{1}{a(s)}\, ds = \infty,$$

$\beta < 1$ (case $a(t) \equiv 1$ by Belohorec 1967)

$$\int_0^\infty b(t) \left(\int_0^t \frac{1}{a(t)} \right)^\mu dt = \infty \quad \mu \in [0, \beta].$$

Extensions: Heidel and Kiguradze (1973), for two-dimensional differential systems by Izyumova and Mirzov (1976-1977), Kordonis-Philos (1998)

Nonoscillation: all solutions of (EF) are nonoscillatory – Atkinson ($\beta > 1$), Heidel ($0 < \beta < 1$).

Equation with the classical Laplacian

Half-linear differential equation

$$\left(a(t)|x'|^{\alpha}\operatorname{sgn} x'\right)' + b(t)|x|^{\alpha}\operatorname{sgn} x = 0,$$

I. Bihari (1957), A. Elbert (1979–2000), T. Kusano (since 1995), O. Došlý (since 1998)

Properties of half-linear equations:

- homogeneity property: if x solution, then kx also solution
- no singular solutions
- Sturmian theory all sols are either oscillatory or nonoscillatory!
- variational method: energy functional

$$\int_{t_1}^{t_2}\left(a(t)|x'|^{p} - b(t)|x|^{p}\right)dt$$

The discrete analogues of differential equations

Sturm-Liouville difference equation

$$x_{n+1} - 2x_n + x_{n-1} + b_n x_n = 0 \quad (n = 1, 2, 3, \dots) \qquad (L\Delta)$$

or using the forward difference operator $\Delta x_n = x_{n+1} - x_n$

$$\Delta^2 x_{n-1} + b_n x_n = 0 \quad (n = 1, 2, 3, \dots) \qquad (L\Delta)$$

A solution x is *oscillatory* if

$$x_n x_{n+1} \leq 0$$

for infinity many n; otherwise it is *nonoscillatory*.
Sturmian theory: all sols are either oscillatory or nonoscillatory

- ↪ P. Hartman, Difference equations: disconjugacy, principal solutions, Green's functions, complete monotonicity (1978)
- ↪ D.B.Hinton, R.T. Lewis, Spectral analysis of second order difference equations (1978)
- ↪ W.T. Patula (1979,1981)

Emden-Fowler difference equation

$$\Delta^2 x_n + b_n |x_{n+1}|^\lambda \operatorname{sgn} x_{n+1} = 0, \quad \lambda \neq 1.$$

Physical applications:

↪ F. Weil (1980): Existence theorem for the difference equation $y_{n+1} - 2y_n + y_{n-1} = h^2 f(y_n)$

↪ R.B.Potts (1981): Exact solution of a difference approximation to Duffing's equation

J. W. Hooker, W.T. Patula (1983):

- Main results are discrete analogues of the Atkinson and Belohorec oscillation criterion.
- Differential equation has all oscillatory solutions bounded but difference equation has unbounded oscillatory solutions!
- Discrete analogue of Atkinson nonoscillation theorem turns out to be false!

Some other nonanalogous properties:

- Discretization process of the differential operator

$$\left(a(t)x'\right)' \quad \rightsquigarrow \quad \Delta\left(a_n \Delta x_n\right), \; a_n > 0.$$

  Oscillation theory in full generality a_n can attain also negative values. Fibonacci recurrence relation

$$x_{n+2} = x_{n+1} + x_n$$

  can be written as

$$\Delta((-1)^n \Delta x_n) + (-1)^n x_{n+1} = 0.$$

- Any solution of difference equations with the classical p-Laplacian can be defined for all $n = 0, 1, 2, \ldots$!

Quasilinear difference equation

$$\Delta(a_n|\Delta x_n|^{\alpha}\text{sgn } \Delta x_n) + b_n|x_{n+1}|^{\beta} \text{ sgn } x_{n+1} = 0, \qquad (1)$$

where $\alpha > 0, \beta > 0$.

Two different cases:

- $\alpha = \beta$: half-linear equation
- $\alpha \neq \beta$: Emden-Fowler type equation

$\alpha = \beta$: O. Došly, J.R. Graef, J. Jaroš (2003), P. Řehak (2000-2008),

Nonlinear difference equations and two-dimensional systems: P. J. Y. Wong, R. P. Agarwal (1995), J. R. Graef (1998), E. Thandapani (2001), W. Jingfa (1997), W.T. Li (2001-2003), S. R. Grace (2005), S. K. Saker (2003),. . .

M. Cecchi, Z.D., M. Marini, I. Vrkoc (2004-2006)

Characteristic integrals

$$\left(a(t)|x'|^{\alpha}\operatorname{sgn}x'\right)' + b(t)|x|^{\beta}\operatorname{sgn}x = 0, \quad \alpha > 0, \ \beta > 0 \qquad \text{(E)}$$

$\alpha > 0$, $\beta > 0$, functions a, b are positive on $[0,\infty)$ and $b \in L^1[0,\infty)$,

$$A(t) := \frac{1}{a^{1/\alpha}(t)}.$$

Characteristic integrals:

$$J_{\alpha} = \int_0^{\infty} A(t)\left(\int_t^{\infty} b(s)\,ds\right)^{1/\alpha} dt,$$

$$K_{\beta} = \int_0^{\infty} b(t)\left(\int_0^{t} A(s)\,ds\right)^{\beta} dt.$$

- (E) has bounded solutions

$$x \sim c \quad \text{as } t \to \infty \quad \Longleftrightarrow \quad J_\alpha < \infty.$$

- (E) has unbounded solutions

$$x \sim c \int_0^t A(s) ds \quad \text{as } t \to \infty \quad \Longleftrightarrow \quad K_\beta < \infty.$$

- Let $\alpha \neq \beta$. Equation (E) has all solutions oscillatory $\Longleftrightarrow$

$$J_\alpha = \infty, \quad K_\beta = \infty.$$

Characteristic integrals:

$$J_{\alpha} = \int_0^\infty A(t) \left(\int_t^\infty b(s)\, ds \right)^{1/\alpha} dt,$$

$$K_{\beta} = \int_0^\infty b(t) \left(\int_0^t A(s)\, ds \right)^{\beta} dt.$$

Fubini theorem: If $\alpha = \beta = 1$, then $J = K$ and so

$$J = \infty \iff K = \infty.$$

Let $\alpha = \beta \neq 1$ or $\alpha \neq \beta \rightsquigarrow$

$$J_{\alpha} = \infty \iff K_{\beta} = \infty \text{ ??} \tag{2}$$

Change of integration

Theorem 1 [Z.D., I. Vrkoč (2004)]

Let A, b be nonnegative integrable functions on $[0, T]$. Define for $x \in [0, T]$

$$K(x) = \int_0^x b(t)\Big(\int_0^t A(s)\, ds\Big)^m dt,$$
$$J(x) = \int_0^x A(t)\Big(\int_t^x b(s)\, ds\Big)^{1/m} dt.$$

(a) If $m \geq 1$ then

$$K(x) \leq J^m(x) \quad \text{for } x \in [0, T].$$

(b) If $0 < m \leq 1$ then

$$J^m(x) \leq K(x) \quad \text{for } x \in [0, T].$$

$$J_{\alpha} = \int_0^{\infty} A(t) \left(\int_t^{\infty} b(s)\, ds \right)^{1/\alpha} dt,$$

$$K_{\beta} = \int_0^{\infty} b(t) \left(\int_0^{t} A(s)\, ds \right)^{\beta} dt.$$

Corollary 1

Let $\alpha = \beta$.
(a) If $\alpha \geq 1$ and $J_\alpha < \infty$, then $K_\alpha < \infty$.
(b) If $\alpha \leq 1$ and $K_\alpha < \infty$, then $J_\alpha < \infty$.

Vice versa does not hold!

Theorem 2 [Cecchi-Z.D.-Marini-Vrkoč (2006)]

Let $\alpha < \beta$. If $K_\beta < \infty$, then $J_\alpha < \infty$.
Let $\alpha > \beta$. If $J_\alpha < \infty$, then $K_\beta < \infty$.

Vice-versa does not hold.

$\rightsquigarrow$ The possible cases are the following:

$$J_\alpha = \infty, \quad K_\beta = \infty$$
$$J_\alpha < \infty, \quad K_\beta < \infty$$
$$J_\alpha = \infty, \quad K_\beta < \infty \quad \text{for } \alpha = \beta > 1 \text{ or } \alpha > \beta$$
$$J_\alpha < \infty, \quad K_\beta = \infty \quad \text{for } \alpha = \beta < 1 \text{ or } \alpha < \beta.$$

Discrete analogue

$$\Delta(a_n|\Delta x_n|^\alpha \text{sgn } \Delta x_n) + b_n|x_{n+1}|^\beta \text{ sgn } x_{n+1} = 0, \qquad (3)$$

where

$$A_n := \frac{1}{a_n^{1/\alpha}}, \quad \sum_{n=0}^{\infty} b_n < \infty.$$

Characteristic series:

$$S_\alpha = \sum_{n=0}^{\infty} A_n \left(\sum_{k=n}^{\infty} b_k \right)^{1/\alpha},$$

$$T_\beta = \sum_{n=0}^{\infty} b_n \left(\sum_{k=0}^{n} A_k \right)^{\beta}.$$

Huo-Li (2003): If $S_\alpha = \infty$ and $T_\beta = \infty$, then Emden-Fowler type equation has all solutions oscillatory.

Change of summation

Theorem 3

Let $\alpha < \beta$ *or* $\alpha = \beta \leq 1$. *Then*

$$T_\beta < \infty \implies S_\alpha < \infty.$$

Let $\alpha > \beta$ *or* $\alpha = \beta \geq 1$. *Then*

$$S_\alpha < \infty \implies T_\beta < \infty.$$

Proofs and applications to difference equations with $b_n < 0$:

- $\alpha = \beta$: Cecchi, Z.D., Marini, Vrkoč - JMAA (2005).
- $\alpha \neq \beta$: Cecchi, Z.D., Marini, Vrkoč - Math. Bohemica (2006), Proceedings 8th ICDEA (2007).

Emden-Fowler differential equation

Asymptotics of Emden-Fowler differential equation

$$x'' + b(t)|x|^{\beta} \operatorname{sgn} x = 0, \quad \beta > 1. \tag{EF}$$

where b is a positive continuous functions for $t \geq 0$.

If x is a sol. of (EF), then $-x$ is a sol. too. So, we will consider positive solutions.

In view of the sign of b, every positive solution is increasing.

Positive solutions can be classified as:

subdominant $\iff$ $x(\infty) = c_x,\ x'(\infty) = 0$ . . . . $\mathbb{M}_B$

intermediate $\iff$ $x(\infty) = \infty,\ x'(\infty) = 0$ . . . . $\mathbb{M}_{\infty 0}$

dominant $\iff$ $x(\infty) = \infty,\ x'(\infty) = d_x$ . . . . $\mathbb{M}_{\infty B}$

c_x, d_x are positive constants.

If x, y and z are subdominant, intermediate and dominant sols, then

$$0 < x(t) < y(t) < z(t) \quad \text{for large } t.$$

$$x'' + b(t)|x|^{\beta} \operatorname{sgn} x = 0, \quad \beta > 1. \tag{EF}$$

Integral conditions:

$$J = \int_0^{\infty} t\, b(t)dt < \infty, \quad K_{\beta} = \int_0^{\infty} t^{\beta}\, b(t)dt = \infty.$$

$$J < \infty,\ K_{\beta} < \infty : \mathbb{M}_B \neq \emptyset,\ \mathbb{M}_{\infty B} \neq \emptyset,\ \mathbb{M}_{\infty 0} = \emptyset$$
$$J < \infty,\ K_{\beta} = \infty : \mathbb{M}_B \neq \emptyset,\ \mathbb{M}_{\infty B} = \emptyset,\ ??\ \mathbb{M}_{\infty 0} \neq \emptyset??$$

Possible cases:
$\rightsquigarrow$ all solutions are nonoscillatory – Naito (2013)
$\rightsquigarrow$ there exist oscillatory solutions – Z.D., M. Marini

Define the functions

$$F(t) = t^{(\beta+3)/2} b(t), \quad \varphi(t) = (F(t))^{1/(\beta-1)} \sqrt{t}.$$

Theorem 4

[Z.D., M. Marini 2014] Let the function F is nondecreasing on $[T, \infty)$, $T > 0$ and

$$\int_0^\infty t\, b(t) dt < \infty, \quad \int_0^\infty t^\beta\, b(t) dt = \infty.$$

If

$$\int_T^\infty b(t) \varphi^\beta(t)\, dt < \infty,$$

then (EF) *has intermediate solutions x such that for $t \geq T$*

$$0 < x(t) \leq c\sqrt{t}.$$

In addition, if b is locally of bounded variation on $[T, \infty)$, then there exist three types of solutions:

- subdominant solution which are positive increasing on (T, ∞),
- intermediate, i.e. $x \to \infty$, $x' \to 0$
- oscillatory

Example 1. Consider Emden-Fowler equation

$$x'' + \frac{1}{4t^{(\beta+3)/2}}|x|^{\beta}\operatorname{sgn} x = 0 \quad \beta > 1, \ (t \geq 1). \qquad (4)$$

We have $F(t) = 1/4$. Thus this equation has intermediate solutions satisfying

$$0 < x(t) \leq c\sqrt{t}, \quad t \in (1, \infty).$$

Obviously,

$$x(t) = \sqrt{t} \quad \text{is such solution.}$$

Moreover, this equation has 3 types of solutions:

- subdominant solution which are positive increasing on $(1, \infty)$
- intermediate , i.e. $x \to \infty$, $x' \to 0$
- oscillatory

M. Cecchi, Z.D., M. Marini (2008–2009)

Consider the half-linear difference equations with the deviating argument

$$\Delta\left(a_n|\Delta x_n|^\alpha \operatorname{sgn} \Delta x_n\right) + b_n|x_{n+q}|^\alpha \operatorname{sgn} x_{n+q} = 0. \tag{H}$$

$q \in \mathbb{Z}$, $\alpha > 0$, $\{a_n\}$, $\{b_n\}$ are positive sequences for $n \geq 0$

$$Y_a = \sum_{n=0}^{\infty} \frac{1}{a_n^{1/\alpha}} = \infty, \quad Y_b = \sum_{n=0}^{\infty} b_n < \infty.$$

All eventually positive solutions are increasing.

- Second order equation: $q = 1$ (Sturm-Liouville theory)
- The advanced argument: $q = 2, 3, 4 \ldots$
- The delay argument: $q = 0, -1, -2, -3, \ldots$

Characteristic series

The convention $\sum_{n_1}^{n_2} u_i = 0$ if $n_1 > n_2$.

$$S_\alpha = \sum_{n=0}^{\infty} \left(\frac{1}{a_n}\right)^{1/\alpha} \left(\sum_{k=n}^{\infty} b_k\right)^{1/\alpha},$$

$$T_\alpha(q) = \sum_{n=0}^{\infty} b_n \left(\sum_{k=0}^{n+q-1} \left(\frac{1}{a_k}\right)^{1/\alpha}\right)^{\alpha}.$$

Compatibility of conditions:

- When $q = 1$: change of summation
- When $q \neq 1$: $T_\alpha(1) \leq T_\alpha(q)$ for $q > 1 \Longrightarrow$
  $S_\alpha = \infty$, $T_\alpha(q) < \infty$ is not possible when $\alpha < 1$, $q > 1$!
  $S_\alpha < \infty$, $T_\alpha(q) = \infty$ is not possible when $\alpha > 1$, $q < 1$!

Nonoscillatory solutions: any eventually positive solution x of (H) is one of the following types:

$$\mathbb{M}^+_{\ell,0} = \{x : \ x(\infty) = c_x, \ x^{[1]}(\infty) = 0\} \qquad \textit{subdominant}$$
$$\mathbb{M}^+_{\infty,0} = \{x : \ x(\infty) = \infty, \ x^{[1]}(\infty) = 0\} \qquad \textit{intermediate}$$
$$\mathbb{M}^+_{\infty,\ell} = \{x : \ x(\infty) = \infty, \ x^{[1]}(\infty) = d_x\} \qquad \textit{dominant}$$

where $x^{[1]}$ is the quasi-difference of solution

$$x_n^{[1]} = a_n|\Delta x_n|^{\alpha}\text{sgn } \Delta x_n.$$

- (H) has subdominant solutions $\iff S_\alpha < \infty$.
- (H) has dominant solutions $\iff T_\alpha(q) < \infty$.

Theorem 5

For (H) we have:

i_1) $S_\alpha < \infty$, $T_\alpha(q) < \infty$. *Then* $\mathbb{M}^+_{\ell,0} \neq \emptyset$, $\mathbb{M}^+_{\infty,0} = \emptyset$, $\mathbb{M}^+_{\infty,\ell} \neq \emptyset$.

i_2) $S_\alpha = \infty$, $T_\alpha(q) < \infty$. *Then* $\mathbb{M}^+_{\ell,0} = \emptyset$, $\mathbb{M}^+_{\infty,0} \neq \emptyset$, $\mathbb{M}^+_{\infty,\ell} \neq \emptyset$.

i_3) $S_\alpha < \infty$, $T_\alpha(q) = \infty$. *In addition, if* $q \neq 1$ *assume*

$$\liminf_n a_n > 0. \tag{5}$$

Then $\mathbb{M}^+_{\ell,0} \neq \emptyset$, $\mathbb{M}^+_{\infty,0} \neq \emptyset$, $\mathbb{M}^+_{\infty,\ell} = \emptyset$.

i_4) $S_\alpha = \infty$, $T_\alpha(1) = \infty$. *Let (H) with* $q = 1$ *be nonoscillatory. In addition, if* $q \neq 1$ *assume* (5). *Then* $\mathbb{M}^+_{\ell,0} = \emptyset$, $\mathbb{M}^+_{\infty,\ell} = \emptyset$ *and*

$$\mathbb{M}^+_{\infty,0} \neq \emptyset.$$

The convergence of the series $T_\alpha(q)$ depends on q:

Example. Let $0 < \alpha < 1$ and define the sequences a, b so that

$$\sum_{i=0}^{k} \left(\frac{1}{a_i}\right)^{1/\alpha} = 2^{k^2}, \quad b_k = 2^{-\alpha k^2} \frac{1}{k+1}.$$

Then

$$T_\alpha(0) = \sum_{k=1}^{\infty} b_k \left(\sum_{i=0}^{k-1} \left(\frac{1}{a_i}\right)^{1/\alpha} \right)^{\alpha} = \sum_{k=1}^{\infty} \frac{1}{k+1} 2^{-\alpha(2k-1)} < \infty,$$

$$T_\alpha(1) = \sum_{k=0}^{\infty} \frac{1}{k+1} 2^{-\alpha k^2}\ 2^{\alpha k^2} = \sum_{k=0}^{\infty} \frac{1}{k+1} = \infty,$$

$$S_\alpha \le \beta^{1/\alpha} \sum_{k=0}^{\infty} 2^{k^2} \left(\frac{1}{k+1}\right)^{1/\alpha} 2^{-k^2} < \infty.$$

Hence

$$S_\alpha < \infty, \quad T_\alpha(q) < \infty \ \text{ for } q \leq 0, \quad T_\alpha(1) = \infty.$$

- $q = 1$ all unbounded solutions of (H) are intermediate
- $q \leq 0$ all unbounded positive solutions of (H) are dominant.

The delay argument changes the growth of unbounded solutions!

Example. Consider the linear equation

$$\Delta^2 x_n + b_n x_{n+1} = 0, \tag{6}$$

and the delay equation

$$\Delta^2 y_n + b_n y_{n-5} = 0, \tag{7}$$

where

$$b_n = 3^{-10n+25}(3^{-4n-4} + 2 \cdot 3^{-2n-1} + 1).$$

We have $S_1 < \infty$, so both equations have subdominant solutions.

- ↪ (6) is nonoscillatory (subdominant and dominant solutions)
- ↪ (7) has bounded positive solution and oscillatory solution $y = \{(-1)^n 3^{-n^2}\}$.

The delay produces oscillatory solutions!

Example. Consider the linear equation

$$\Delta(a_n \Delta x_n) + b_n x_{n+1} = 0 \tag{8}$$

and the delay equation

$$\Delta(a_n \Delta y_n) + b_n y_{n-\tau} = 0, \quad \tau \in \mathbb{N}, \tag{9}$$

where

$$a_0 = 1,\ a_{n+1} = 2^{-n^2}\left(2^{(2n+1)} - 1\right)^{-1}\ (n \ge 0), \quad b_n = 2^{-n^2+2n}.$$

Then $S_1 = \infty$, $T_1(1) = \infty$ and $T_1(-1) < \infty$.

↪ (8) has all solutions oscillatory

↪ (9) has nonoscillatory (intermediate and dominant) solutions.

The delay produces nonoscillatory solutions!

M. Cecchi, Z.D., M. Marini (2009–2013)

Consider **equation with bounded phi-Laplacian**

$$\big(a(t)\Phi(x')\big)' + b(t)F(x) = 0, \quad (t \geq 0) \qquad (E_\Phi)$$

where:

(i) Φ is an increasing odd homeomorphismus

$$\Phi : \mathbb{R} \to (-\sigma, \sigma),$$

$0 < \sigma < \infty$, $\Phi(u)u > 0$ for $u \neq 0$;

(ii) F is a real continuous increasing function on $\mathbb{R}$ such that $F(u)u > 0$ for $u \neq 0$;

(iii) $a, b \colon [0, \infty) \to (0, \infty)$ are continuous functions and

$$\int_0^\infty b(t)\, dt < \infty\,.$$

Prototypes are PDE's involving the mean curvature operator

$$\operatorname{div}(\Phi(\nabla u)) = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$$

Mean curvature operator: $\Phi_C : \mathbb{R} \to (-1, 1)$

$$\Phi_C(u) = \frac{u}{\sqrt{1+|u|^2}}$$

The lack of the homogeneity property $\Phi(uv) = \Phi(u)\Phi(v)$!

Equation with p-Laplacian $\hookrightarrow$ as a two-dimensional system

$$x' = \frac{1}{\Phi_p^*(a(t))}\Phi_p^*(y) , \qquad y' = -b(t)|x|^{\beta} \operatorname{sgn} x$$

$\Phi_p^*(u) = |u|^{1/\alpha} \operatorname{sgn} u.$

$\hookrightarrow$ Consider the integral

$$I_\lambda = \int_0^\infty \Phi^* \left(\frac{\lambda}{a(t)} \right) dt$$

where Φ^* is the inverse function of Φ and

$$\Lambda = \cap_{t \geq 0} \big(0, \sigma\, a(t) \big).$$

Two different cases when $\sigma < \infty$:

- $\liminf_{t \to \infty} a(t) = 0$: $\Lambda = \emptyset$.
- $\liminf_{t \to \infty} a(t) > 0$: Λ is a bounded nonempty interval. *Leighton type oscillation criterion*, ...

Theorem 6

Assume $\liminf_{t\to\infty} a(t) = 0$.

i$_1$) *If*

$$\limsup_{t\to\infty} \frac{1}{a(t)} \int_t^\infty b(s)\,ds = \infty\,,$$

then any continuable solution of (E_Φ) *is oscillatory.*

i$_2$) *If*

$$\limsup_{t\to\infty} \frac{1}{a(t)} \int_t^\infty b(s)\,ds < \infty \tag{10}$$

and there exists $\overline{\mu} > 0$ *such that*

$$\int^\infty \Phi^*\left(\overline{\mu}\,\frac{1}{a(t)} \int_t^\infty b(s)\,ds\right)\,dt < \infty\,,$$

then (E_Φ) *has nonoscillatory sols.* $x \sim L$ *for any* $L > 0$ *sufficiently small.*

Discrete analogue

Difference equation with bounded Φ Laplacian

$$\Delta(a_n\Phi(\Delta x_n)) + b_n|x_{n+1}|^\gamma \operatorname{sgn} x_{n+1} = 0, \qquad (D_\Phi)$$

where $\gamma > 0$, $\gamma \neq 1$, $\Phi : \mathbb{R} \to (-\sigma, \sigma)$, $\sigma < \infty$ and

$$\sum_{n=0}^{\infty} \frac{1}{a_n} = \infty, \quad \sum_{n=0}^{\infty} b_n < \infty.$$

Prototype: $\Phi_C : \mathbb{R} \to (-1, 1)$ (Mean curvature operator)

Solutions of (D_Φ) satisfying $x_{n_0} = c$, $x_{n_0+1} = d$ $(n_0 \geq 0)$ need not exist for arbitrary c, d!

Theorem 7

Assume

$$\limsup_n \frac{1}{a_n} \sum_{k=n}^{\infty} b_k > 0. \tag{11}$$

Then (D_Φ) *is oscillatory.*

Condition (11) $\Longrightarrow$ $\liminf_{t\to\infty} a(t) = 0$.

Theorem 7 does not have a continuous counterpart!

If $\widetilde{a}, \widetilde{b}$ are continuous positive functions for $t \geq t_0$ such that

$$\int_{t_0}^{\infty} \widetilde{b}(t)dt < \infty,$$

$$0 = \liminf_{t\to\infty} \frac{1}{\widetilde{a}(t)} \int_t^{\infty} \widetilde{b}(s)ds < \limsup_{t\to\infty} \frac{1}{\widetilde{a}(t)} \int_t^{\infty} \widetilde{b}(s)ds,$$

then the differential equation

$$\big((\widetilde{a}(t)\Phi(x'(t))\big)' + \widetilde{b}(t)|x(t)|^{\gamma}\text{sgn } x(t) = 0$$

can have by Theorem 6 nonoscillatory solutions!

Consider

$$\Delta(a_n\Phi(\Delta x_n)) + b_n|x_{n+1}|^{\gamma}\text{sgn } x_{n+1} = 0, \qquad (D_\Phi)$$

where

$$\lim_{u\to 0}\frac{\Phi(u)}{u} = c \quad (0 < c < \infty). \qquad (12)$$

$$\Phi_C(u) = \frac{u}{\sqrt{1+|u|^2}} \quad \ldots \quad c = 1$$

Comparison with the Emden-Fowler equation

$$\Delta(a_n\Delta y_n) + b_n|y_{n+1}|^{\gamma}\text{ sgn } y_{n+1} = 0 \quad \gamma \neq 1. \qquad (13)$$

Theorem 8

Let $\gamma > 1$. Equation (D_Φ) is oscillatory if and only if Emden-Fowler equation (13) is oscillatory, i.e.

$$\sum_{n=0}^{\infty} \frac{1}{a_n} \sum_{k=n}^{\infty} b_k = \infty.$$

References

- Cecchi M., Došlá Z., Marini M., Vrkoč I.: *Integral conditions for nonoscillation of second order nonlinear differential equations*, Nonlinear Anal. **64** (2006).
- Došlá Z., Vrkoč I.: *On extension of the Fubini theorem and its application to the second order differential equations*, Nonlinear Anal. **57** (2004).
- Cecchi M., Došlá Z., Marini M.:, *Oscillation of a class of differential equations with generalized Phi-Laplacian,* Proc. Royal. Soc. Edin. (2013)
- Došlá Z., Marini M.:, *On super-linear Emden-Fowler type differential equations*, J. Math. Anal. Appl. (2014).
- Kordonis I.G.E., Philos Ch.G.: *On the oscillation of nonlinear two-dimensional differential systems*, Proc. Amer. Math. Soc. **126** (1998).

References II

- KUSANO T., NAITO Y.: *Oscillation and nonoscillation criteria for second order quasilinear differential equations,* Acta Math. Hungar. **76** (1997), 81-99.
- MIRZOV J.D.: Asymptotic properties of solutions of the systems of nonlinear nonautonomous ordinary differential equations, (Russian), Maikop, Adygeja Publ. 1993. English translation: Folia, Mathematics **14**, Masaryk University Brno 2004.
- ROGOVCHENKO Y.V., TUNCAY F.: *Oscillation criteria for second-order nonlinear differential equations with damping*, Nonlinear Anal. **69** (2008), 208-221.
- SUGIE J., YAMAOKA N.: *Growth conditions for oscillation of nonlinear differential equations with p-Laplacian*, J. Math. Anal. Appl. **306** (2005), 18-34.
- YAMAOKA N.: *Oscillation criteria for second-order damped nonlinear differential equations with p-Laplacian*, J. Math. Anal. Appl. **325** (2007), 932-948.

References III

- Agarwal R.P, Bohner M., Grace S.R., 'Regan D.: Discrete Oscillation Theory, Hindawi Publ. Corp., New York , 2005.
- Cecchi M., Došlá Z., Marini M., I. Vrkoč: *Summation inequalities and half-linear difference equations*, J. Math. Anal. Appl., **302**, 2005, 1-13.
- Cecchi M., Došlá Z., Marini M.: *On the growth of nonoscillatory solutions for difference equations with deviating argument*, Adv. Difference Equ. (2008)
- J. Jiang, X. Li: *Oscillation and nonoscillation of two-dimensional difference systems*, J. Computat. Appl. Math., **188**, (2006), 77-88.
- H.F. Huo, W.T. Li: *Oscillation of certain two-dimensional nonlinear difference systems*, Comp. Math. Appl., **45**, (2003), 1221-1226.
- W.T. Li: *Classification Schemes for Nonoscillatory Solutions of Two-Dimensional Nonlinear Difference Systems*, Computers Math. Appl. **42,** (2001), 341–355.
- P.J.Y. Wong, R.P. Agarwal: *Oscillation and monotone solutions of second order quasilinear difference equations*, Funkc. Ekvacioj, **39,** (1996), 491-507.

Thank you for your attention!